Numer. Math. Theor. Meth. Appl. Vol. **10**, No. 4, pp. 737-759
doi: 10.4208/nmtma.2016.0001 November 2017

Blowup of Volterra Integro-Differential Equations and Applications to Semi-Linear Volterra Diffusion Equations

Zhanwen Yang[1], Tao Tang[2] and Jiwei Zhang[3,*]

[1] *Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China*
[2] *Department of Mathematics, Southern University of Science and Technology, Shenzhen, Guangdong 518055, China*
[3] *Beijing Computational Science Research Center, Zhongguancun Software Park II, Haidian District, Beijing 100094, China*

Received 27 December 2016; Accepted (in revised version) 31 March 2017

Abstract. In this paper, we discuss the blowup of Volterra integro-differential equations (VIDEs) with a dissipative linear term. To overcome the fluctuation of solutions, we establish a Razumikhin-type theorem to verify the unboundedness of solutions. We also introduce leaving-times and arriving-times for the estimation of the spending-times of solutions to ∞. Based on these two typical techniques, the blowup and global existence of solutions to VIDEs with local and global integrable kernels are presented. As applications, the critical exponents of semi-linear Volterra diffusion equations (SLVDEs) on bounded domains with constant kernel are generalized to SLVDEs on bounded domains and $\mathbb{R}^N$ with some local integrable kernels. Moreover, the critical exponents of SLVDEs on both bounded domains and the unbounded domain $\mathbb{R}^N$ are investigated for global integrable kernels.

AMS subject classifications: 35K55, 45K05
Key words: Volterra integro-differential equations, volterra diffusion equations, blowup, global existence, razumikhin theorem.

1. Introduction

Volterra integral equations (VIEs) have a lot of applications in physics and experimental sciences: problems in mechanics, scattering theory, spectroscopy, stereology, seismology, elasticity theory, plasma physics (see in [25]). The blowup of solutions to VIEs rises in [25] and a particular example is given in [27]. After that a sequence of blowup results for some special VIEs are done by [22, 26, 32] (see also in the survey papers [17, 33] and the references in [6]).

*Corresponding author. *Email addresses:* yangzhan_wen@126.com (Z. Yang), tangt@sustc.edu.cn (T. Tang), jwzhang@csrc.ac.cn (J. Zhang)

In [6], we present two features of blowup solutions : (i) the tendency to ∞, (ii) the finite spending-time for tending to ∞. Based on some assumptions ensuring the strictly monotone increasing of solutions to VIEs, the necessary and sufficient conditions for the blowup of Hammerstein-type nonlinear VIEs are discovered. By transforming into the equivalent form of VIEs, we also present some blowup results of Volterra integro-differential equations (VIDEs)

$$u'(t) = -\lambda u(t) + \int_0^t k(t-s)u^p(s)\mathrm{d}s, \quad t > 0, \tag{1.1a}$$

$$u(0) = u_0 > 0. \tag{1.1b}$$

While the condition $\lambda \leqslant 0$ is imposed in our paper [6] for the strict monotonicity of solutions. The blowup of solutions to non-homogeneous VIDEs is discussed in [21] under some conditions on the non-homogeneous term such that the solution is also strictly increasing. The blowup results of VIDEs with a dissipative linear term (i.e. $\lambda > 0$) and constant kernels are discussed in [35], since the solutions are convex and eventually increasing. Up to now, for (1.1) with $\lambda > 0$ and a general kernel, the blowup of fluctuation solutions is still open.

As important as VIDEs, semi-linear Volterra diffusion equations (SLVDEs)

$$u_t = \Delta u + \int_0^t k(t-s)u^p(s,x)\mathrm{d}s, \quad t > 0,\ x \in \Omega \subseteq \mathbb{R}^N, \tag{1.2a}$$

$$u(0,x) = u_0(x) \gneqq 0, \qquad x \in \Omega, \tag{1.2b}$$

$$u(t,x) \equiv 0, \qquad x \in \partial\Omega \tag{1.2c}$$

are introduced to model the effects of the memory effects in a population dynamics in [38, 39], and widely used in compression of poro-viscoelastic media in [11], the thermodynamics of phase transition in [3], reaction-diffusion problems in [9], and the theory of nuclear reactor kinetics in [15, 28–30]. The finite blowup analysis to (1.2) is begun in [16] and a complete result is obtained in [35] that the critical exponent of SLVDEs on bounded domains with constant kernels is $p^* = \infty$. That is to say, any positive solution to (1.2) blows up in finite time. However, for any $p > 1$, there always exists a global positive solution to

$$u_t = \Delta u + u^p(t,x), \quad t > 0,\ \ x \in \Omega, \tag{1.3}$$

when Ω is a bounded domain (see in [18, 23, 24]). Hence the critical exponent $p^* = 1$ of (1.3) is totally changed by the non-local time-integration with a constant kernel. Note that the local problem (1.3) can be written as the form of (1.2) with a Dirac delta function $k(z) = \delta(z)$ and the difference between constant kernels and delta functions is the integration on the whole interval $[0, \infty)$. Therefore it is more interesting how the blowup of SLVDEs on bounded domains is influenced by the global integrability of kernels. It is also interesting whether the critical exponent of SLVDEs on $\mathbb{R}^N$ is also influenced by the kernels.

By Kaplan's first eigenvalue and eigenfunction, the blowup results of SLVDEs on bounded domains come from the ones of solutions to (1.1), but the linear coefficient λ corresponding to the first eigenvalue of the Laplacian operator is positive. The blowup analysis