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A Weak Galerkin Finite Element Method for the Navier-Stokes Equations

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Abstract. In this paper, a weak Galerkin finite element method (WGFEM) is proposed for solving the Navier-Stokes equations (NSEs). The existence and uniqueness of the WGFEM solution of NSEs are established. The WGFEM provides very accurate numerical approximations for both the velocity field and pressure field, even with very high Reynolds numbers. The salient feature is that the flexibility of the WGFEM for the choice of the order of the velocity and pressure comparing to the standard finite element methods. Optimal order error estimates in both $|\cdot|_{\mathbf{W}_h}$ and L^2 norms are proved for the semi-discretized scheme. Numerical simulations are presented to show the efficiency of the WGFEM.

AMS subject classifications: 65N15, 65N30, 65M12, 76D07

Key words: Weak Galerkin, finite element method, the Navier-Stokes equations.

1 Introduction

The simulation of fluid dynamics is essential in many applications in science and engineering, such as the physics of complex flows, hydrodynamics of submarines, the forecasting of ocean dynamics, and so on. It is well known that numerical methods for the Navier-Stokes equations (NSE) are indispensable for better understanding of fluid dynamics, in particular for possible chaos or turbulent behavior. In recent years, with the development of computing power of supercomputers, the importance of numerical methods for large-scale fluid dynamics problems is increasingly more significant.

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Thus far, numerical approaches based on finite difference method (FDM), finite element method (FEM), finite volume method (FVM), spectral method, discontinuous Galerkin (DG) method etc., have been developed for NSEs and we refer to [5, 9, 11, 14, 19, 21, 39], and references therein for some of these results.

The FEM is one of the most popular and versatile numerical methods. However, several issues arise when the standard FEM is used to solve NSEs. Complex phenomena such as boundary layers, turbulence, etc., are governed by NSEs with high Reynolds numbers. The corresponding numerical simulations are highly challenging and require fine spatial meshes in order to guarantee accuracy in the spatial direction, which leads to a highly nonlinear system with huge degree of freedom (DOF) to be solved [14,23]. Many numerical methods are proposed to resolve this issue. Based on the Oseen iteration and error correction strategy, Wang and Wong proposed an error correction method for solving NSEs with high Reynolds numbers [33]. In [17], Greengard and Kropinsko rewrote the NSEs with Reynolds numbers around 3000 as nonlinear fourth-order partial differential equations (PDEs) satisfied by the stream function, and then used an inexact Newton method to solve the nonlinear PDEs at each iteration rapidly. In [8, 29], Qiu and his collaborators propose a hybridizable discontinuous Galerkin method for the NSEs, where a superconvergence property of the velocity lead to an elementwise postprocessed approximate velocity and the computational costs are reduced sharply. Although the NSEs at high Reynolds numbers have been studied for the last decades [1, 14, 15], it is still highly urgent to boost the capability of numerical solvers.

As is well known, if the finite element spaces for the velocity and pressure fields don't match, numerical results will suffer severely from instabilities or highly oscillatory pressure field. There are two strategies to avoid this phenomena: the mixed FEMs and the stabilized FEM. In the former case, the DOF in the finite element space for the pressure field is always chosen as one order lower than that for the velocity field such that the inf-sup condition [27] is satisfied. In the latter case, the Galerkin least-squares (GLS) stabilization is adopted [12], where the stabilizing terms are given by minimizing the squared residual of the momentum equation over each element.

The choice of temporal discretization of NSEs is also technical. Among many time stepping techniques, such as implicit method [30], semi-implicit method [16], alternating direction implicit method [26], and so on, the projection method is possibly the most popular temporal approach [10, 18, 32]. By adopting this splitting technique, the velocity and pressure fields are decoupled, which avoids solving a saddle point problem at each time step. For ease of exposition, we focus on the spatial discretization, and adopt Crank-Nicolson scheme for temporal discretization in the sequel, and refer interested readers to [10] and the references therein for more details of temporal discretization.

This paper concerns with the numerical approximation of the Navier-Stokes equations by the weak Galerkin finite element method (WGFEM or WGM). The weak Galerkin method was first proposed by Wang and Ye to deal with second-order elliptic problems [33]. Comparing with the standard FEM, the WGM does not have to satisfy any inter-element continuity constraints and has assorted applications in the physics or engi-