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# THE SINGULARITY-SEPARATED METHOD FOR THE SINGULAR PERTURBATION PROBLEMS IN 1-D

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**Abstract.** The singularity-separated method(SSM) for the singular perturbation problem $-\epsilon u''+bu'+cu=f(x), u(0)=u(1)=0$, is proposed for the first time. The solution is expressed as $u=w-v$, where $w$ is the solution of corresponding third boundary value problem and $v$ is an exact singular function. We have proved a global regularity, $||w||_2 \leq C$, where the constant $C$ is independent of $\epsilon$, and discussed three kinds of finite element (FE) methods with SSM. Numerical results show that these FE-solutions have the high accuracy when only one element in boundary layer is taken.

**Key words.** Singular perturbation problem, singularity-separated method, third boundary value, second order regularity, finite elements.

## 1. Introduction

L.Prandtl in 1904 found that the speed of fluid moved in a pipeline would decrease acutely along the wall, which was called boundary layer phenomenon. This phenomenon is common in many practical fields. Mathematically, if the coefficient of the highest order derivative term in a differential equation is a small parameter, its solution often has a boundary layer. It is very difficult to simulate numerically the singular perturbation solution. In this paper we will propose a new idea to deal with this difficulty.

We consider one-dimensional singular perturbation problem

$$\begin{cases} Au=-\epsilon u''+bu'+cu=f(x) \quad in\ J=[0,1] \\ u(0)=0, \quad u(1)=0\ or\ u'(1)=0 \end{cases} \tag{1}$$

where $\epsilon$ is a small parameter, e.g., $\epsilon=10^{-3}\sim 10^{-10}$, constants $b>0, c>0$. The solution has the singularity $e^{b(x-1)/\epsilon}$ in the boundary layer near $x=1$. Denote by $\tau=p_0\epsilon|\ln\epsilon|/b$ the width of boundary layer. Early the finite difference method(FDM) was used and its error in boundary layer vibrates strongly [11]. For this situation G.Shishkin[10, 12] proposed a famous Shishkin-mesh, i.e., $J$ is divided into two subintervals $J_0=(0,1-\tau)$ and $J_1=(1-\tau,1)$, in which the smooth subinterval $J_0$ is subdivided into N-uniform meshes with step-size $h=(1-\tau)/N\approx 1/N$, and the boundary layer $J_1$ is subdivided into N-uniform meshes with the much smaller step-size $h'=\tau/N<<h$. The convergence of FDM under the Shishkin mesh was studied [1, 10, 12, 13]. Besides, the finite element method (FEM) was also discussed [4, 8]. The Shishkin-mesh is successful to simulate the singular perturbation problems up to now.

We are interesting in (local) discontinuous Galerkin finite element methods(LDG) much more. It is well-known that LDG can simulate the acute change of singular solution very well [18]. By introducing a new variable $q=u'$, the original equation

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(1) becomes a first order elliptic system

$$\begin{cases} -\epsilon q' + bq + cu = f(x), \\ u' - q = 0, \\ u(0) = 0, \quad u(1) = 0 \ or \ q(1) = 0, \end{cases} \tag{2}$$

Celiker-Cockburn [2] discussed the highest order superconvergence for LDG solution when $\epsilon = 1$. At the same time Xie-Zhang [15, 16, 17] independently studied LDG method on Shishkin meshes for the singular perturbation problem, and the vibration of error was weakened greatly. They got the uniform convergence and superconvergence independent of $\epsilon$.

The accuracy of numerical approximation depends not only on the number of the grid nodes, but also on the parameter $\epsilon$. The smaller $\epsilon$ is, the thinner the boundary layer is, and the more acutely the solution varies. We feel that Shishkin meshes have some defects for high dimensional problems. For example, more and more nodes concentrate in the boundary layer of domain so that the geometrical aspect ratios of the meshes become very large, even the meshes near corner are extremely small, thus many troubles will appear in FE-approximation, such as the accuracy, efficiency, stability and so on. In numerical simulation, we gradually realize that it is necessary to improve the mathematical expression of singular perturbation solution. Numerical algorithms strongly depend on the regularity estimates of the solution, and constructing various singular functions is a fundamental method to study the regularity. They are systematically summarized in monograph [11], and the singular functions in 1-D are also discussed in it. Generally speaking, these singular functions are applied to study the regularity, but (maybe due to their inexactness) they have not been taken as a correct function to construct a new high-performance algorithm.

So we propose a new idea, called singularity-separated method(SSM). We decompose the original problem into two sub-problems as follows. Firstly introduce an auxiliary third boundary problem, whose solution has weaker singularity and the free term $f$ is eliminated. Secondly construct a singular correct function exactly, which is a special solution with homogenous free term $f = 0$, and is a part of numerical solution we need. It should be emphasized that these solutions of two sub-problems have the weaker singularity, which are directly and simply applied in FE-computation. In particular only one or two elements in boundary layer are needed, thus the local refinement of Shishkin mesh is not necessary. This is the main aim of us. Although only one-dimensional case is discussed in this paper, the theoretical and computational framework proposed are valid for multi-dimensional singularly perturbed elliptic and convection-diffusion problems, which need some new analysis techniques yet.

We also have another motivation. The solutions $u$ of fluid dynamics problems with viscosity, such as Navier-Stokes equation, have classical energy estimate $\|u\|_1 \leq \|f\|/\epsilon^2$. For small $\epsilon$, the bound becomes awfully large, unless $f$ is also small. Thus the essential difficulty will occur in studying the solvability by the fixed point principle and numerical simulations. We want to know that whether SSM can solve the trouble $f$. This is an expectation in future.

An outline of this article is as follows. In Section 2, we present the singularity separated method and state two results which provide the theoretical basis of SSM. The proofs of those two theorems are carried out in details in Section 3. In Section 4, we present three kinds of FEMs with SSM and the numerical experiments provided show the robustness of SSM.