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Extending GCR Algorithm for the Least Squares Solutions on a Class of Sylvester Matrix Equations

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Abstract. The purpose of this paper is to derive the generalized conjugate residual (GCR) algorithm for finding the least squares solution on a class of Sylvester matrix equations. We prove that if the system is inconsistent, the least squares solution can be obtained within finite iterative steps in the absence of round-off errors. Furthermore, we provide a method for choosing the initial matrix to obtain the minimum norm least squares solution of the problem. Finally, we give some numerical examples to illustrate the performance of GCR algorithm.

AMS subject classifications: 65F10

Key words: Sylvester matrix equation, Least squares solution, Generalized conjugate residual algorithm, Numerical experiments

1. Introduction

Matrix equations appear frequently in many areas of applied mathematics and play important roles in many applications, such as control theory and system theory [25–27]. For example, the descriptor linear system

$$A_1\dot{x}+A_1\dot{x}+B_0u=0 \tag{1.1}$$

captures the dynamic behavior of many physical systems in practice [29–31] and the second order linear system

$$A_2\ddot{x}+A_1\dot{x}+A_1\dot{x}+B_0u=0 \tag{1.2}$$

has wide applications in vibration and structural analysis, robotics control and spacecraft control [32, 33]. It is known that certain control problems, such as pole/eigenstructure assignment and observer design are closely related to the generalized Sylvester matrix equations (1.1) and (1.2). To solve the additive decomposition problem of a transfer

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matrix [34], we need to find the solution pair (X,Y) of the generalized coupled Sylvester matrix equations

$$\begin{cases} AX - YB = C, \\ DX - YE = F. \end{cases} \tag{1.3}$$

In this paper, we consider the solution of the following matrix equations

$$\begin{cases} A_1XB_1 = C_1, \\ A_2XB_2 = C_2, \end{cases} \tag{1.4}$$

where $A_1 \in R^{p\times m}$, $B_1 \in R^{n\times q}$, $C_1 \in R^{p\times q}$ and $A_2 \in R^{r\times m}$, $B_2 \in R^{n\times s}$, $C_2 \in R^{r\times s}$ are given matrices, and $X \in R^{m\times n}$ is an unknown matrix to be determined.

There have been many papers considering various solutions of the matrix equations(1.4). For instance, Mitra [1, 2] gave conditions for the existence of a solution and a representation of the general common solution to the system (1.4). Navarra et al. [3] derived the sufficient and necessary conditions for the existence of a common solution to the system (1.4). Yuan [4] obtained an analytical expression of the least squares solution of the system (1.4) by using the generalized singular value decomposition (GSVD) of matrices. Sheng and Chen [5] presented a finite iterative method when the system (1.4) is consistent. Cai and Chen [6] constructed an iterative algorithm for the least squares bisymmetric solution of the matrix equations (1.4) by applying the theory of convex analysis. In [24], Dehghan and Hajarian presented an algorithm for solving matrix equations (1.4) in order to obtain (R,S)-symmetric and (R,S)-skew symmetric solution. An efficient iterative method was proposed for finding the generalized centro-symmetric solution of the matrix equations (1.4) by Dehghan and Hajarian [43]. Chen et al. [38] obtained common symmetric least squares solutions of the matrix equations (1.4) by using the LSQR iterative method. Wang et al. [42] presented a direct method to solve the least squares Hermitian problem of the complex matrix equations $(AXB, CXD) = (E,F)$ with the help of matrix-vector product and the Moore-Penrose generalized inverse.

In the past decades, most of the proposed iterative algorithms for solving linear matrix equations were obtained from the extension of algorithms which were previously introduced for solving the linear system of equations $Ax = b$. See for [7–13, 39, 40]. For example, Bai proposed a Hermitian and skew-Hermitian splitting (HSS) iteration algorithm to solve the Sylvester matrix equation

$$AX + XB = F, \tag{1.5}$$

with non-Hermitian and positive definite/semi-definite matrices [44]. A nested splitting conjugate gradient (NSCG) iteration method [45] was proposed for solving the matrix equation

$$AXB = C. \tag{1.6}$$