East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.201216.230317a

Vol. **7**, No. 3, pp. 455-481
August 2017

Dynamics of an Innovation Diffusion Model with Time Delay

Rakesh Kumar[1,*], Anuj K. Sharma[2] and Kulbhushan Agnihotri[3]

[1] *Department of Applied Sciences, SBS State Technical Campus, Ferozepur-152004, Punjab, India.*
[2] *Department of Mathematics, LRDAV College, Jagraon, Ludhiana-142026, Punjab, India.*
[3] *Department of Applied Sciences, SBS State Technical Campus, Ferozepur-152004, Punjab, India.*

Received 20 December 2016; Accepted (in revised version) 23 March 2017.

Abstract. A nonlinear mathematical model for innovation diffusion is proposed. The system of ordinary differential equations incorporates variable external influences (the cumulative density of marketing efforts), variable internal influences (the cumulative density of word of mouth) and a logistically growing human population (the variable potential consumers). The change in population density is due to various demographic processes such as intrinsic growth rate, emigration, death rate etc. Thus the problem involves two dynamic variables viz. a non-adopter population density and an adopter population density. The model is analysed qualitatively using the stability theory of differential equations, with the help of the corresponding characteristic equation of the system. The interior equilibrium point can be stable for all time delays to a critical value, beyond which the system becomes unstable and a Hopf bifurcation occurs at a second critical value. Employing normal form theory and a centre manifold theorem applicable to functional differential equations, we derive some explicit formulas determining the stability, the direction and other properties of the bifurcating periodic solutions. Our numerical simulations show that the system behaviour can become extremely complicated as the time delay increases, with a stable interior equilibrium point leading to a limit cycle with one local maximum and minimum per cycle (Hopf bifurcation), then limit cycles with more local maxima and minima per cycle, and finally chaotic solutions.

AMS subject classifications: 34C23, 34D20, 37L10, 49Q12, 92D25

Key words: Innovation diffusion model, evaluation period, stability analysis, Hopf bifurcation, normal form theory, centre manifold theorem.

*Corresponding author. *Email addresses:* `keshav20070@gmail.com` (R. Kumar), `anujsumati@gmail.com` (A. K. Sharma), `agnihotri69@gmail.com` (K. Agnihotri)

1. Introduction

Since the earliest work in the 1960s [1, 2], there have been many investigations on the spread of new consumer durables. Various aspects in the mathematical modelling have been considered, but the Bass marketing innovation diffusion model [3] is still the foundation for new hypotheses to gain insight into the marketing of innovations. The Bass model assumes that the market is fixed, and the market penetration is described by

$$\frac{dA(t)}{dt} = p\big[m - A(t)\big] + \frac{q}{m}A(t)\big[m - A(t)\big],$$

where A(t) is the number of adopters at time t. The parameter p accounts for new adopters entering the market due to external influences (e.g. the activities of firms in the market, advertising, and attractiveness of the innovation), and the parameter q is the coefficient of imitation refering to the rate of influence of adopters on non-adopters. Refs. [4–13] illustrate the wide use of the Bass model in understanding the spread of new products, and the underlying phenomena responsible for the diffusion of new technology (market innovation) or new products in the market.

In a book, Rogers [14] summarised previous research and concluded that innovation diffusion consists of five steps viz. awareness (an individual is exposed to innovation), interest (individual seeks more information), evaluation (individual applies innovation to his or her situation), trial (individual uses innovation on a small scale), and adoption (the individual makes full use of the innovation). Further, through behavioural research he found that these five steps can be simplified into a two-step flow process i.e. media influences innovative opinion leaders to adopt a new product, who in turn influence other people to adopt the product [14–18]. In particular, Mahajan *et al.* [16] discussed earlier contributions in the management and marketing literature to cumulative understanding of innovation diffusion dynamics.

Models with time delay to exhibit product evaluation stages have been proposed by many researchers [19–22]. One mathematical model describes the dynamics for users of one product in two different patches [23]. Stability of a competitive innovation diffusion model in a market [24, 25] and global stability of an innovation diffusion model for n products [26] have been discussed. An innovation diffusion model with nonlinear acceptance describes the dynamics of three competing products in a market [27], and a binomial innovation diffusion model for a variable size market with demographic processes of entrance or exit from market compartments has been considered [28]. Shukla *et al.* [29] proposed that the innovation diffusion process is affected by variable external influences such as advertisements and the change of density of the non-adopter population due to intrinsic growth rate, emigration or death rate *inter alia*. The model analysis shows that the adopter population density increases as the parameters related to the growth rate of the non-adopter population as the rate of external influences increase, and the main effect of the cumulative density of the variable external influences is to make the equilibrium level of the adopter population density reach its equilibrium at a much faster rate. Wang *et al.* [22] considered delay to describe the process of evaluation and decision-making, whereas Shukla *et al.* [29]