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# Perturbation Bounds and Condition Numbers for a Complex Indefinite Linear Algebraic System

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**Abstract.** We consider perturbation bounds and condition numbers for a complex indefinite linear algebraic system, which is of interest in science and engineering. Some existing results are improved, and illustrative numerical examples are provided.

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**Key words**: Complex indefinite linear algebraic system, perturbation bounds, condition numbers.

## 1. Introduction

Complex indefinite linear algebraic systems arise in many scientific computing and engineering applications, including structural dynamics [1,6,12], electromagnetism [11], wave propagation [5,10], electrical power system modelling and the discretisation of different types of Helmholtz equation [7,8,9]. Many authors have recently considered iterative methods for solving complex indefinite linear algebraic systems, including the Modified HSS (MHSS) and preconditioned Modified HSS (PMHSS) methods [2,3,4].

The complex indefinite linear algebraic system considered is

$$\mathscr{A}x \equiv (W+iT)x = b \,, \tag{1.1}$$

where $i=\sqrt{-1}$, $W$ and $T \in \mathbb{R}^{(m+n)\times(m+n)}$, $x$ and $b \in \mathbb{C}^{(m+n)}$, and we assume the coefficient matrix $\mathscr{A}$ is nonsingular. One case eample of this complex indefinite linear algebraic system is the finite difference discretisation of the Helmholtz partial differential equation

$$-\nabla\cdot(c\nabla\mu)-\sigma_1\mu+i\sigma_2\mu=f \,, \tag{1.2}$$

where the coefficients $c$, $\sigma_1$ and $\sigma_2$ are real-valued functions. Discretising Eq. (1.2) with Dirichlet boundary conditions and both $\sigma_1$ and $\sigma_2$ strictly positive produces Eq. (1.1) with an indefinite real part $W$ and a definite imaginary part $T$ of the discrete operator $A=$

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$W+iT$. Another application concerns the interior penalty discontinuous Galerkin (IPDG) approximation of the prototypical Helmholtz problem

$$-\triangle\mu - k^2\mu = f \ in\, \Omega := \Omega_1 \setminus D\,, \tag{1.3}$$

$$\mu = 0 \ \text{ on } \ \Gamma_D\,, \tag{1.4}$$

$$\frac{\partial \mu}{\partial n_\Omega} + ik\mu = g \ \text{ on } \ \Gamma_R\,. \tag{1.5}$$

Here $i=\sqrt{-1}$, $k \in R$ is a given positive large number (wave number), $\Omega_1 \subset R^d$, $d=2,3$ is a polygonal/polyhedral domain (often a d-rectangle), $D \subset \Omega_1$ represents a scatterer, $\Gamma_R := \partial D$ hence $\partial\Omega = \Gamma_R \cup \Gamma_D$, and $n_\Omega$ is the unit outward normal to $\partial\Omega$. The Robin boundary condition (1.3) is known as the first order absorbing boundary condition, and implies that the scatterer is sound-soft. The finite element method has also been widely used to discretise Helmholtz problems involving various boundary conditions.

We focus on how the iterative solution of the complex indefinite linear algebraic system (1.1) changes when the coefficient matrix of the complex indefinite linear algebraic system undergoes a small perturbation — and in particular, we discuss perturbation bounds and condition numbers that to the best of our knowledge have not been analysed previously. The system (1.1) can be rewritten in the 2-by-2 block form

$$\mathscr{A}x \equiv \left[\begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix} + i\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{1.6}$$

where $W_1, T_1 \in \mathbb{R}^{m\times m}$; $W_4, T_4 \in \mathbb{R}^{n\times n}$; $W_2, T_2 \in \mathbb{R}^{m\times n}$; $W_3, T_3 \in \mathbb{R}^{n\times m}$ and $x_1, b_1 \in \mathbb{C}^m$; $x_2, b_2 \in \mathbb{C}^n$. The problem can be specified as follows: $W$, $T$, $b$ in Eq. (1.1) or $W_i$, $T_i$ $(i=1,2,3,4)$ and $b_j$ $(j=1,2)$ in the corresponding Eq. (1.6) are subjected to small perturbations denoted by $\Delta W$, $\Delta T$, $\Delta b$, $\Delta W_i$, $\Delta T_i$ and $\Delta b_j$, respectively. We consider perturbation bounds denoted by $\|\Delta x\|_2$, where $\Delta x$ is the error in the solution $x$. The relevant perturbed system respectively obtained from Eq. (1.1) or Eq. (1.6) is

$$(\mathscr{A}+\Delta\mathscr{A})(x+\Delta x) = (W+\Delta W+iT+i\Delta T)(x+\Delta x) = b+\Delta b$$

or

$$\begin{aligned}&\left[\begin{pmatrix} W_1+\Delta W_1 & W_2+\Delta W_2 \\ W_3+\Delta W_3 & W_4+\Delta W_4 \end{pmatrix} + i\begin{pmatrix} T_1+\Delta T_1 & T_2+\Delta T_2 \\ T_3+\Delta T_3 & T_4+\Delta T_4 \end{pmatrix}\right]\begin{pmatrix} x_1+\Delta x_1 \\ x_2+\Delta x_2 \end{pmatrix} \\ =& b+\Delta b \\ =& \begin{pmatrix} b_1+\Delta b_1 \\ b_2+\Delta b_2 \end{pmatrix},\end{aligned} \tag{1.7}$$

where $\Delta W_1, \Delta T_1 \in \mathbb{R}^{m\times m}$, $\Delta W_4, \Delta T_4 \in \mathbb{R}^{n\times n}$, $\Delta W_2, \Delta T_2 \in \mathbb{R}^{m\times n}$, $\Delta W_3, \Delta T_3 \in \mathbb{R}^{m\times n}$, $\Delta x_1, \Delta b_1 \in \mathbb{C}^m$ and $\Delta x_2, \Delta b_2 \in \mathbb{C}^n$.

Some definitions, notation and useful lemmas to deduce the main results are given in Section 2. Perturbation bounds for the complex linear algebraic system are derived in Section 3, and the corresponding condition numbers are discussed in Section 4. We present some illustrative numerical examples in Section 5, and briefly summarise our results in Section 6.