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Perturbation Bound for the Eigenvalues of a Singular Diagonalizable Matrix

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Abstract. In this short note, we present a sharp upper bound for the perturbation of eigenvalues of a singular diagonalizable matrix given by Stanley C. Eisenstat [3].

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1. Introduction

For $A \in \mathbb{C}^{n\times n}$, the smallest nonnegative integer k satisfying the rank equation,

$$rank(A^k) = rank(A^{k+1})$$

is called the index of the matrix A [1,9]. If $X \in \mathbb{C}^{n\times n}$ is the unique solution of the three matrix equations

$$A^{k+1}X = A^k, \qquad XAX = X, \qquad AX = XA\,,$$

we call X the Drazin inverse A^D. If $index(A) = 1$, then the Drazin inverse is reduced to the group inverse denoted by $A^\sharp$ [1,9].

Let us now recall the classical Bauer-Fike theorem of 1960 and its version from 1999.

Theorem 1.1. (Bauer-Fike Theorem [2,4]) *Let A be diagonalizable — i.e. $A = X\Lambda X^{-1}$, where the diagonal matrix $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$, λ_i is the eigenvalue of A. Let E be the perturbation of A and μ the eigenvalue of $A + E$. Then*

$$\min_i \left|\lambda_i - \mu\right| \le \kappa_2(X)\,\|E\|_2\,. \tag{1.1}$$

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If A is invertible, then

$$\min_i \left|\frac{\lambda_i - \mu}{\lambda_i}\right| \le \kappa_2(X)\left\|A^{-1}E\right\|_2, \tag{1.2}$$

where $\kappa_2(X) = \|X^{-1}\|_2\|X\|_2$ is the condition number of X with respect to the 2-norm.

Wei *et al.* [7, 8] explored how to extend the classical Bauer-Fike theorem to include the singular case, with the help of the group inverse. Later, Eisenstat [3] gave a different version as follows:

Theorem 1.2. *Suppose that A is singular diagonalizable — i.e. $A = X\begin{pmatrix}\Lambda_1 & \\ & \mathbf{0}\end{pmatrix}X^{-1}$, where $\Lambda_1 = diag(\lambda_1, \lambda_2, \cdots, \lambda_r)$, λ_i $(i = 1, 2, \cdots, r)$ is the nonzero eigenvalue of A. Let E be the perturbation of A, and μ the eigenvalue of $A+E$. If $|\mu| > \kappa_2(X)\|E\|_2$, then*

$$\min_i \left|\frac{\lambda_i - \mu}{\lambda_i}\right| \le \sqrt{1+\alpha^2}\kappa_2(X)\left\|A^{\sharp}E\right\|_2, \tag{1.3}$$

where $\alpha = \kappa_2(X)\|E\|_2/\sqrt{|\mu|^2 - (\kappa_2(X)\|E\|_2)^2}$.

2. Main Results

In this section, we present our main result that improves the upper bound of Ref. [3].

Theorem 2.1. *Assume that A is singular diagonalizable and E is the perturbation of A, and μ is the eigenvalue of $A+E$. If $|\mu| > \|X^{-1}(I - AA^{\sharp})EX\|_2$. Then*

$$\min_i \left|\frac{\lambda_i - \mu}{\lambda_i}\right| \le \sqrt{1+\beta^2}\left\|X^{-1}A^{\sharp}EX\right\|_2, \tag{2.1}$$

where $\beta = \|X^{-1}(I - AA^{\sharp})EX\|_2/\sqrt{|\,\mu\,|^2 - \|X^{-1}(I - AA^{\sharp})EX\|_2^2}$.

Proof. Let $A = X\begin{pmatrix}\Lambda_1 & \\ & \mathbf{0}\end{pmatrix}X^{-1}$, where $\Lambda_1 = diag(\lambda_1, \lambda_2, \cdots, \lambda_r)$ is a nonsingular diagonal matrix. Let x be an eigenvector of $A+E$ associated with μ, and denote

$$X^{-1}EX = \begin{pmatrix}E_{11} & E_{12}\\ E_{21} & E_{22}\end{pmatrix} \quad \text{and} \quad X^{-1}x = \begin{pmatrix}x_1\\ x_2\end{pmatrix}.$$

Since $\mu x = (A+E)x$,

$$\mu\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \mu X^{-1}x = X^{-1}(A+E)XX^{-1}x = \begin{pmatrix}E_{11}+\Lambda_1 & E_{12}\\ E_{21} & E_{22}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix},$$