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On Perturbation Bounds for the Joint Stationary Distribution of Multivariate Markov Chain Models

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Abstract. Multivariate Markov chain models have previously been proposed in for studying dependent multiple categorical data sequences. For a given multivariate Markov chain model, an important problem is to study its joint stationary distribution. In this paper, we use two techniques to present some perturbation bounds for the joint stationary distribution vector of a multivariate Markov chain with s categorical sequences. Numerical examples demonstrate the stability of the model and the effectiveness of our perturbation bounds.

AMS subject classifications: 65M10, 78A48

Key words: Multivariate Markov chain models, stationary distribution vector, condition number, relative bound.

1. Introduction and Notations

In many real world problems, there are situations where one would like to consider a number of Markov chains $\{\mathbf{X}_{t,i}\}_{i=1}^{s}$ together at the same time, particularly in the analysis of multiple categorical data sequences. The state of the i-th chain $\mathbf{X}_{t+1,i}$ at time $(t+1)$ often depends not only on $X_{t,i}$ but also on $\{\mathbf{X}_{t,1},\dots,\mathbf{X}_{t,i-1},\mathbf{X}_{t,i+1},\dots,\mathbf{X}_{t,s}\}$, resulting in a multivariate Markov chain model. In a conventional model where the multivariate Markov chain has s chains and each chain has the same set of m states, the total number of states is $O(m^s)$. Consequently, one needs to develop simplified multivariate Markov chain models that can capture both the inter-relations and intra-relations among the given chains with a relatively low number of model parameters. A multivariate Markov chain model was proposed for this purpose in Ref. [2], and applied to demand forecasting. Ref. [3] provides

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a detailed survey of multivariate Markov chain models. The purpose of this paper is to propose some perturbation bounds on the joint stationary distribution vector for multivariate Markov chain models. To consider the stability of the joint probability distribution of a multivariate Markov chain, we need to analyse the change of the joint distribution under a small perturbation of the transition matrix, and there are many results on the perturbation theories of Markov chains.

Let us denote the transition probability matrix of a finite irreducible homogeneous Markov chain by P. The stationary distribution vector of P is the unique positive vector $\boldsymbol{\pi}$ satisfying $\boldsymbol{\pi} = P\boldsymbol{\pi}$ and $\sum_j \pi_j = 1$. Suppose that the matrix P is perturbed to the matrix $\tilde{P}$, the transition probability matrix of another finite irreducible homogeneous Markov chain. On denoting the stationary distribution vector of $\tilde{P}$ by $\tilde{\boldsymbol{\pi}}$, the goal is to describe the change $\tilde{\boldsymbol{\pi}} - \boldsymbol{\pi}$ in the stationary distribution in terms of the change $E \equiv \tilde{P} - P$ in the transition probability matrix. For some vector norms, we have

$$||\tilde{\boldsymbol{\pi}} - \boldsymbol{\pi}|| \leq \kappa ||E||$$

for various different condition numbers κ — e.g. see [5, 7, 8, 10, 11, 13, 15, 16]. However, to the best of our knowledge there is no discussion on perturbation theory for multivariate Markov chain models.

In this paper, we analyse the effects of a small perturbation to the joint stationary distributions of a finite irreducible multivariate Markov chain, when Q is the joint transition probability matrix of such a multivariate Markov chain and

$$\mathbf{\Pi} = (\boldsymbol{\pi}^{(1)T}, \boldsymbol{\pi}^{(2)T}, \ldots, \boldsymbol{\pi}^{(s)T})^T$$

is the joint stationary distribution vector satisfying

$$Q\mathbf{\Pi} = \mathbf{\Pi} \quad \text{and} \quad \sum_{i=1}^{m} [\boldsymbol{\pi}^{(j)}]_i = 1, \quad 1 \leq j \leq s\,.$$

Our goal is to describe the effect on $\tilde{\mathbf{\Pi}}$ when Q is perturbed by a matrix E such that

$$\tilde{Q} = Q + E$$

is the joint transition probability matrix of another irreducible multivariate Markov chain. We first propose perturbation bounds for the joint stationary distribution of a multivariate Markov chain. This is particularly important because the model parameters are different when different estimation methods are employed. Some condition numbers and interesting numerical measures will also be provided. However, while it is theoretically possible to compute condition numbers κ, it is usually expensive — and another possibility is to propose a relative bound that is easy to find without computing $\mathbf{\Pi}$.

The following notation is used throughout this paper:

- for any $\xi \in \mathbb{C}^N$, ξ_i denotes the ith element;
- $\mathbf{1}_l = (1, 1, \ldots, 1)^T$ and $\mathbf{0}_l = (0, 0, \ldots, 0)^T$ are column vectors with dimension l; and