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The Hamiltonian Field Theory of the Von Mises Wave Equation: Analytical and Computational Issues

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Abstract. The Von Mises quasi-linear second order wave equation, which completely describes an irrotational, compressible and barotropic classical perfect fluid, can be derived from a nontrivial least action principle for the velocity scalar potential only, in contrast to existing analog formulations which are expressed in terms of coupled density and velocity fields. In this article, the classical Hamiltonian field theory specifically associated to such an equation is developed in the polytropic case and numerically verified in a simplified situation. The existence of such a mathematical structure suggests new theoretical schemes possibly useful for performing numerical integrations of fluid dynamical equations. Moreover it justifies possible new functional forms for Lagrangian densities and associated Hamiltonian functions in other theoretical classical physics contexts.

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1 Introduction

The dynamics of non viscous barotropic fluids is characterized by well defined Lagrangian [1] as well as Hamiltonian classical field theory formulations [1–6] developed in terms of coupled density and velocity fields. For general rotational configurations moreover, a geometrically sound scheme exists coupling the so called 'Clebsch velocity potentials' to density [1,7]. Such a theory, after linearization, leads to the Analogue Gravity framework

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[8–35] in which part of the fluid linear perturbations behaves kinematically as a massless scalar field described by a wave equation in a curved space-time, leading to the concept of analogue systems as the acoustic black holes for instance. For irrotational classical flows, it has been shown by the authors [36, 37] that the Analogue Geometry formalism is codified at exact level in the Von Mises second order quasi-linear wave equation (VME) [38]. While in standard fluid dynamics works, density and velocity get coupled via the continuity and momentum balance equations, the main advantage of the VME is to be a decoupled quasi-linear second order wave equation for the velocity potential only. We point out that at linear level the Analogue Gravity formalism applies also to quantum fluids as described by the Gross-Pitaevskii equation in hydrodynamical form [1]. Unfortunately in this case the analogy with General Relativity works at first order perturbative level only [8]. It is well known that decoupling of nonlinear partial differential equation systems is not a trivial task. For instance, in Quantum Physics, the simplest system, i.e. the free particle described by complex Schroedinger equation for the wave function ψ (of whom the Gross-Pitaevskii mathematically represents a generalization by including an additional non linear term $\propto|\psi|^2\psi$) is:

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi, \tag{1.1}$$

where $\hbar=h/2\pi$ with h being the Planck's constant and $\vec{\nabla}$ standing for the gradient operator so that ∇^2 is the Laplacian. By using the well known Madelung hydrodynamical representation [1] for the wave function i.e. $\psi=Ae^{iB}$, the above equation splits into the two real equations

$$\hbar\frac{\partial B}{\partial t}-\frac{\hbar^2}{2m}\frac{\nabla^2 A}{A}+\frac{\hbar^2}{2m}(\vec{\nabla}B)^2=0, \tag{1.2a}$$

$$2\frac{\partial A}{\partial t}+2\frac{\hbar}{m}\vec{\nabla}A\cdot\vec{\nabla}B+\frac{\hbar A}{m}\nabla^2 B=0, \tag{1.2b}$$

which mimic a fluid dynamical problem with mass density $\rho=mA^2$ and velocity $\vec{v}=\vec{\nabla}\Phi=\frac{\hbar}{m}\vec{\nabla}B$ where Φ is a velocity potential. With manipulations we can cast the above relations into a continuity and a generalized Bernoulli equations:

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})=0,\qquad \frac{\partial\Phi}{\partial t}+\frac{1}{2}(\vec{\nabla}\Phi)^2+C=0 \tag{1.3}$$

including the additional quantum term $C=-(\hbar^2/2m^2)(\nabla^2 A/A)$. Unfortunately this quantum fluid problem because of the quantity C cannot be simply decoupled into higher order equations for A and B alone. On the other hand, in the classical perfect irrotational fluid case described by Euler's and continuity equations, assuming a barotropic equation of state which links pressure p and density ρ, the quantum term C in the above relations is replaced by the enthalpy $\int dp/\rho$ and, specifically for a polytropic equation of state, the decoupling into a quasilinear wave equation for the velocity potential only (the VME