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Karhunen-Loève Expansion for the Discrete Transient Heat Transfer Equation

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Abstract. In this paper we perform and analyze a Karhunen-Loève expansion on the solution of a discrete heat equation. Unlike the continuous case, several choices can be made from the numerical scheme to the numerical time integration. We analyze some of these choices and compare them. In literature, it is shown that the KL-expansion's error depends on the singular values of the matrix (or the operator) which we attempt to compress, but there is few results on the decay of these singular values. The core of this article is to prove the exponential decay of the singular values. To achieve this result, the analysis is conducted in a classical way by considering the spatial correlation of the temperature. And then, we analyze the problem in a more general view by using the Krylov matrices with Hermitian argument, which are a generalization of the well-known Vandermonde matrices. Some computations are made using MATLAB to ensure the performance of the Karhunen-Loève expansion. This article presents an application of this work to an identification problem where the data are disturbed by a Gaussian white noise.

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Key words: Karhunen-Loève expansion, discrete temperature field.

1 Introduction

In many applications, we compute the same partial differential equation with different parameters. For instance, let us assume that we want to find the more adequate material to our target application. In order to do so, we run a simulation of the PDE in which the material is represented by one or several parameters, changing for each computation these parameters. However, computing the solution each time is costly. We could construct a basis with a much smaller dimension than with a regular method, by choosing solutions with special parameters. The reduction of the dimension is theoretically ensured by the exponential decay of the Kolmogorov N-width, for linear PDEs and some

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non-linear PDEs (see [29]). This result allows us to find the solution to our PDE in a space of dimension N such that the error between the solution of this equation and the approximate solution computed with the reduced basis decreases fastly with N (see [6], [16] for more details). Two main methods are usually used : the *greedy method* which consist in constructing a hierarchical basis provided by an estimate error (see [15]), and the *Proper Orthogonal Decomposition (POD)* which compresses a set of snapshot (solutions to the PDE for fixed parameters) to withhold the main information (see [16], [17]). For parametric PDEs, we can show that the performance of the *greedy method* is related to the Kolmogorov N-width (see [16]).

In this article, we focus on Time-dependent equations, for which a POD basis can be constructed. The idea remains the same : provided some snapshots (which can be solutions at either different time steps or different space points), we perform a *Singular Value Decomposition (SVD)* on the correlation matrix. This compression method appears in many fields under various names such as *Principal Components Analysis (PCA)* in data analysis (see [14]), or *Karhunen-Loève expansion* in statistics to compress a stochastic process ([3]). The error estimation of the POD in the literature shows that the performance of this method is related to the decay of the eigenvalues of the correlation matrix ([25] or [26]). For experimental temperature fields, a POD was performed and analyzed in [27]. However, there is no proof on the decay rate of the eigenvalues. In [28], they explicit this decay by using the smoothness of the solution. Nevertheless, in [1] for bi-variate functions, and in [5] for the heat transfer equation, by using some algebraic properties satisfied by the spatial correlation matrix, they obtain an exponential decay.

For anisotropic parameters or in dimension greater than 1, the analytic solution of the PDE is not accessible. Based on the analysis made in the continuous case, we construct a Karhunen-Loève expansion for a discrete solution and show the efficiency of the method by analyzing the decay of the eigenvalues.

We first construct the Karhunen-Loève expansion for a discrete solution. We then focus in Section 2 on the transient heat transfer equation discretized in time by two well-known numerical schemes. By expanding the discrete temperature field in the basis form by the eigenvectors of the Laplace operator, we give an estimate on the error between the temperature field and the truncated KL-expansion. We adopt a new point a view in Section 3, by using the algebraic properties satisfied by the Krylov matrices with Hermitian arguments. Section 4 is dedicated to numerical results : some computations of the eigenvalues to support the theoretical analysis made in Section 2, and an application of this work to the identification of parameters when the data are disturbed by a Gaussian white noise.

1.1 The Karhunen-Loève expansion for discrete solution

Let us consider a discrete temperature field in time T^n, represented by a vector of size N_t such that each component belongs to $L^2(\Omega)$. For each step time, the only assumption on the function $T^n(\mathbf{x})$ is to belong to L^2 for the space variable. Yet, when we solve a PDE or