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A Fast Shift-Splitting Method for Singular Generalized Saddle Point Problems

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Abstract. For the iteration solution of singular generalized saddle point problems, a fast shift-splitting iteration method based on shift-splitting technique and symmetric and skew-symmetric splitting with respect to the upper-left block of the system matrix is proposed in this paper. Semi-convergence of the proposed method is carefully studied for singular case, and the conditions guaranteeing the semi-convergence are derived. Numerical experiments of a class of linearized Navier-Stokes equations are implemented to demonstrate the feasibility and effectiveness of the proposed method.

AMS subject classifications: 65F10, 65F50

Key words: Shift-splitting iteration method, singular generalized saddle point problems, semi-convergence.

1 Introduction

Consider the solution of systems of linear equations with the following block 2×2 structure

$$\begin{bmatrix} A & B^T \\ -B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \text{ or } \mathcal{A}u=b, \tag{1.1}$$

where $A\in\mathbb{R}^{n\times n}$ is a nonsymmetric positive definite matrix, $B\in\mathbb{R}^{m\times n}$ is a rectangular matrix with $m\le n$, $C\in\mathbb{R}^{m\times m}$ is a symmetric positive semi-definite matrix, $f\in\mathbb{R}^n$ and $g\in\mathbb{R}^m$ are given vectors.

The generalized saddle point problems, i.e., the systems of the form (1.1) arise in a variety of scientific computing and engineering applications, including computational fluid

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dynamics [7], mixed finite element approximation of elliptic partial differential equations [17], weighted and equality constrained least squares estimation [9], inversion of geophysical data [20] and others.

If the matrix B is of full column rank, i.e., $\mathrm{rank}(B)=m\leq n$, the generalized saddle point matrix $\mathcal{A}$ is nonsingular. If the matrix B is rank deficient and $\mathrm{null}(C)\cap\mathrm{null}(B^T)\neq\{0\}$, the generalized saddle point matrix $\mathcal{A}$ is singular. Here, $\mathrm{null}(\cdot)$ denotes the null space of the corresponding matrix. In this work, we are particularly interested in the latter. And, for the singular case, we suppose that b is in the range of $\mathcal{A}$, that is, the singular saddle point problems are consistent in this paper.

In many cases A,B and C are large sparse matrices and iterative techniques are preferable for solving (1.1). In recent years, many effective methods have been proposed for solving singular saddle point problems in the literature, for example, the Uzawa-type methods [5,26,27], Hermitian and skew-Hermitian splitting type methods [1–3,14,21,22], and Krylov subspace methods [18,25]. Parameterized Uzawa method was studied in [27], and the semi-convergence of this method was proved when it was applied to solve the singular saddle point problems. Minimum residual and conjugate gradient methods were proposed for solving the rank-deficient saddle point problems in [18, 25], respectively. Inexact Uzawa method, which covers the Uzawa method, the preconditioned Uzawa method, and the parameterized method as special cases, was discussed for singular saddle point problems in [26], and the semi-convergence result under restrictions was proved by verifying two necessary and sufficient conditions. Moreover, sufficient conditions for the semi-convergence of several Uzawa-type methods were also provided in [26].

In this paper, we construct a fast shift-splitting iteration method for singular generalized saddle point problems based on the ideas of the shift-splitting iteration method [6, 12] and the Hermitian and skew-Hermitian splitting technique [4, 21, 28]. The idea of shift-splitting iteration method was first proposed by Bai, Yin and Su in [6] for solving a class of non-Hermitian positive definite linear systems. Then it was extended by Cao, Du and Niu in [11] to solve saddle point problems, and it was generalized by Salkuyeh for saddle point problems in [23]. After that, for nonsymmetric saddle point problems, Cao and Miao in [13] proposed the generalized shift-splitting (GSS) method. Recently, Shen and Shi applied the GSS iteration method to solve a broad class of nonsingular and singular generalized saddle point problems in [24]. In this paper, a fast shift-splitting iteration method is studied. Semi-convergence of this method for singular case is carefully analyzed. Numerical experiments further show that the proposed method is efficient and feasible.

The rest of this paper is organized as follows. In Section 2, a fast shift-splitting iteration method for singular generalized saddle point problems is established. In Section 3, the semi-convergence of the proposed method for singular case is studied. Numerical experiments are presented in Section 4. Finally, a brief conclusion is given in Section 5.