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A New Family of Difference Schemes for Space Fractional Advection Diffusion Equation

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Abstract. The second order weighted and shifted Grünwald difference (WSGD) operators are developed in [Tian, Zhou and Deng, Math. Comput., 84 (2015), pp. 1703–1727] to solve space fractional partial differential equations. Along this direction, we further design a new family of second order WSGD operators; by properly choosing the weighted parameters, they can be effectively used to discretize space (Riemann-Liouville) fractional derivatives. Based on the new second order WSGD operators, we derive a family of difference schemes for the space fractional advection diffusion equation. By von Neumann stability analysis, it is proved that the obtained schemes are unconditionally stable. Finally, extensive numerical experiments are performed to demonstrate the performance of the schemes and confirm the convergence orders.

AMS subject classifications: 26A33, 65M06, 65M12

Key words: Riemann-Liouville fractional derivative, WSGD operator, fractional advection diffusion equation, finite difference approximation, stability.

1 Introduction

As an extension of classical calculus, fractional calculus also has more than three centuries of history. However, it seems that the applications of fractional calculus to physics and engineering attract enough attention only in the last few decades. Nowadays, it has been found that a broad range of non-classical phenomena in the applied sciences and engineering can be well described by some fractional kinetic equations [17]. The fractional calculus is becoming more and more popular, especially in describing the anomalous diffusions [2,16,17,22], which arise in physics, chemistry, biology and other complex dynamics. With the appearance of various kinds of fractional partial differential equations

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(PDEs), finding the ways to effectively solve them becomes a natural topic. Based on the integral transformations, sometimes we can find the analytical solutions of linear fractional PDEs with constant coefficients [17, 18, 22]. Even in these cases, most of the time their solutions are expressed by transcendental functions or infinite series. So looking for the numerical solutions of fractional PDEs becomes a realistic expectation in practical applications; and designing efficient and robust numerical schemes for fractional PDEs is the basis to this task.

Over the past decade, the finite difference methods are implemented in simulating the space fractional advection diffusion equations [6, 10, 12, 13, 15, 24]. In recent years, high order discretizations and fast methods for space fractional PDEs with Riemann-Liouville fractional derivatives attract many authors' interests. Based on the Toeplitz-like structure of the difference matrix, Wang et al. [32] solve the algebraic equation corresponding to fractional diffusion equation with a $N\log^2 N$ cost; Pang and Sun [20] propose a multigrid method to solve the discretized system of the fractional diffusion equation. By using the linear spline approximation, Sousa and Li provide a second order discretization for the Riemann-Liouville fractional derivatives and establish an unconditionally stable finite difference method for one-dimensional fractional diffusion equation in [25]; the results on two-dimensional two-sided space fractional convection diffusion equation in finite domain can be seen in [5]. Ortigueira [19] gives the "fractional centered derivative" to approximate the Riesz fractional derivative with second order accuracy and this scheme is used by Çelik and Duman in [3] to approximate fractional diffusion equation with the Riesz fractional derivative in a finite domain. For more application and extension of "fractional centered derivative" one can see the recent works [23, 31] and the references therein.

More recently, Tian et al. [30] propose the second order difference approximations, called WSGD approximations, to the Riemann-Liouville space fractional derivatives on the finite domain. Baeumer et al. [1] discuss a higher order Grünwald-type approximations in the infinite domain with the help of semigroup generators. The basic idea of the WSGD approximation is to cancel the low order terms by combining the Grünwald difference operators with different shifts and weights. Along this direction, this paper further introduces a new family of WSGD opertors by providing more to be chosen parameters, called second order WSGD operators II. As specific applications, the second order WSGD operators II are used to discretize the space fractional derivative of the space fractional advection diffusion equation. And the von Neumann stability analyses are performed to the obtained numerical schemes. We theoretically prove and numerically verify that with proper chosen parameters for the WSGD space discretizations, both the obtained implicit and Crank-Nicolson schemes are unconditionally von Neumann stable.

The outline of this paper is organized as follows. In Section 2, we first introduce a new family of second order discretizations of the Riemann-Liouville fractional derivatives. Then two kinds of difference schemes for the one dimensional space fractional advection diffusion equation are presented in Section 3 and the corresponding numerical stabilities are performed. Furthermore, in Section 4, the numerical schemes and the