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A Novel Low-Dimensional Method for Analytically Solving Partial Differential Equations

Jie Sha[1], Lixiang Zhang[1] and Chuijie Wu[2,*]

[1] *Department of Engineering Mechanics, Kunming University of Science and Technology, Kunming 650500, Yunnan, China*
[2] *State Key Laboratory of Structural Analysis for Industrial Equipment School of Aeronautics and Astronautics, Dalian University of Technology, Dalian 116024, China*

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Abstract. This paper is concerned with a low-dimensional dynamical system model for analytically solving partial differential equations (PDEs). The model proposed is based on a posterior optimal truncated weighted residue (POT-WR) method, by which an infinite dimensional PDE is optimally truncated and analytically solved in required condition of accuracy. To end that, a POT-WR condition for PDE under consideration is used as a dynamically optimal control criterion with the solving process. A set of bases needs to be constructed without any reference database in order to establish a space to describe low-dimensional dynamical system that is required. The Lagrangian multiplier is introduced to release the constraints due to the Galerkin projection, and a penalty function is also employed to remove the orthogonal constraints. According to the extreme principle, a set of ordinary differential equations is thus obtained by taking the variational operation of the generalized optimal function. A conjugate gradient algorithm by FORTRAN code is developed to solve the ordinary differential equations. The two examples of one-dimensional heat transfer equation and nonlinear Burgers' equation show that the analytical results on the method proposed are good agreement with the numerical simulations and analytical solutions in references, and the dominant characteristics of the dynamics are well captured in case of few bases used only.

AMS subject classifications: 65M10, 78A48

Key words: Low-dimensional system model, partial differential equation, analytical solution, posterior optimal truncated method.

1 Introduction

Over the past few decades, scientists and engineers were considerable interest in looking for an order-reduced method to effectively model partial differential equations (PDEs).

*Corresponding author.
Email: shajie9981@gmail.com (J. Sha), zlxzcc@126.com (L. X. Zhang), cjwudut@dlut.edu.cn (C. J. Wu)

One of the crucial study motivations is to seek a way to analytically solve reduction order models of a high dimensionality system, while it is generally solved by using a mesh-based discretization technique to investigate the constitutive equation of a physical problem. An analytical solution for a linear low-dimensional problem is ever expected. Unfortunately, the expectation seems to be impossible to most high-dimensional systems, because the nonlinear systems such as high-dimensionality may have quite complex dynamical behaviors, ever possibly evolve into chaos. Therefore, it is extremely valuable if a reduction dimensionality method can be built well to analytically analyze high order dynamical systems.

At the present stage, there are several analytical methods well developed based on the *posterior* and *prior* reduction dimensionality techniques, respectively. Locally linear embedding (LLE) [1] method approximates the local geometry by linear coefficients that are used to reconstruct data points in neighbors, in which the reconstruction errors are controlled by the squared distances between reconstruction data points. The crux of ISOMAP [2] preserves the geodesic manifold distances between all pairs of the data points. As a powerful tool, the proper orthogonal decomposition (POD), firstly introduced by Lumley for studying long-term behavior of turbulence, is beyond reasonable doubt an outstanding *posterior* reduced method. A set of bases is obtained by using POD with a database of experiments or numerical simulations, and these bases on POD are mathematically proved to be optimum in terms of energy. Other attractive property of POD is linear processing while minimizing the average squared distance between the original configuration space and the reduced linear one. Consequently, POD become quite useful to reduce order of a system by construction of new bases on the information under investigative objects [3–6]. Also, POD is valuable to be used by conjunction with other techniques [7–9], for example, Galerkin projection method, to well predict complex dynamics of an airfoil induced by unsteady transonic flow. On the other hand, POD is developed as an order-reduced model with multiple parameters [10,11], by which unstable phenomenon [12] in numerical simulation is effectively suppressed, and computing algorithm is more efficient [13–16]. However, it is noted that a notable shortcoming of the POD methods is highly dependent on *prior* data to construct optimal bases. Thus, the enormous computational cost of numerical simulation, and the time-consuming work of experiment, as well as obtaining usable *prior* data become a challenge in application. So, it is quite necessary to look for a way to circumvent the drawbacks of the *posterior* methods.

Approaching on truncated series expansion is expected to be a good way to overcome the shortcoming of the *posterior* methods, for example, the order-reduced methods on Laguerre polynomials [17]. The methods on other orthonormal polynomials, such as Fourier polynomials, are universal, but it is hard to mathematically deal with boundary conditions. In order to solve the problems of boundary conditions, scientists develop measure methodologies, for example, the proper generalized decomposition (PGD) proposed by Ammar, which is able to well treat multidimensional problems. PGD separates variables by defining a tensor product of unknown approximation basis, and then car-