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Spectral Method Approximation of Flow Optimal Control Problems with H^1-Norm State Constraint

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Abstract. In this paper, we consider an optimal control problem governed by Stokes equations with H^1-norm state constraint. The control problem is approximated by spectral method, which provides very accurate approximation with a relatively small number of unknowns. Choosing appropriate basis functions leads to discrete system with sparse matrices. We first present the optimality conditions of the exact and the discrete optimal control systems, then derive both a priori and a posteriori error estimates. Finally, an illustrative numerical experiment indicates that the proposed method is competitive, and the estimator can indicate the errors very well.

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Key words: Optimal control, state constraint, stokes equations, legendre polynomials, spectral method.

1. Introduction

Flow control problems can be met frequently in many engineering applications; for instance, leading the velocity to a desired one, minimizing turbulence in a flow, enhancing or deterring mixing and designing the optimal shapes to minimize drag in flows for many industrial devices such as aircraft wings, automobile shapes and boats, etc. Fast and efficient numerical methods play an important role in successful applications of flow control problems. Finite element method has been widely used in solving control problems. One can find a systematic introduction to the theoretical analysis and finite element approximation for control problems including flow control problems in, for example, [16, 20, 23, 24, 27, 28]. However, most of these results focus on control-constrained cases. Recently, control problems with state constraints have been widely discussed. On the theoretical aspects, we refer to [2, 14], and have to mention the works of Casas (see, e.g., [6, 7]). In respect of numerical approximation, many studies have been carried out

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to examine finite element analysis for this class of control problems, including point-wise constraint, integral constraint, and L^2-norm constraint (see, e.g., [1,12,19,21,22,25,31]). To compute control problems with state constraints, many numerical strategies are proposed. [3] takes an augmented Lagrangian method to solve state and control constrained problems. Primal-dual active set algorithm is used to solve state constrained problems in [4]. A semi-smooth Newton method is adopted for approximating the regularized point-wise state-constrained optimal control problems in [15]. In [17], the optimal control problem is reformulated to a constrained minimization problem involving only the state, which is characterized by a fourth order variational inequality, then a mixed variational scheme is proposed.

Employing global polynomials as the trial functions, the spectral method has been successfully applied in numerical solutions of PDEs, especially in the field of computational fluid dynamics (see, for example, [8–10, 30], and the references therein). Generally, the solutions to the optimal control problems have limited regularity due to, e.g., the constraints, which leads to the lose of spectral accuracy. However, spectral method enjoys the great superiorities of high-precision and fast convergence rate when the approximated solutions have higher regularity, which is vital to efficient approximation of optimal control problems. Therefore, the spectral method has gained increasing popularity in solving control problems governed by PDEs in the past several years. In [13], spectral method is used to approximate the elliptic control problems with integral control constraint, both a priori and a posteriori error estimates are derived, and numerical tests confirm the efficiency of the proposed method. In [32], a priori error estimates are derived for control problems with integral state constraint. Generally, control problems with state constraints are far more complicated to analyze than control-constrained ones. To our best knowledge, there has been a lack of discussion on the spectral approximation of flow control problems with H^1-norm state constraint, though this class of control problems can be applied in many practical considerations. In fact, for many casting products made by pouring molten metals into mold cavities, the velocity and its derivative of the molten metals should be constrained to ensure the quality of the products. Moreover, it is better to restrict the velocity and its derivative of the flow that passing through the corner. In these cases, we can try to use the control models with H^1-norm constraints for flow velocity.

Let $\Omega \subset \mathbb{R}^2$ be a rectangular domain, C denotes a general positive constant independent of N, the order of the spectral approximation. We shortly describe the structure of our paper as follows: In Section 2, a priori error estimates are derived for Galerkin spectral approximation of the flow control problem. Then, a posteriori error estimates are proposed in Section 3. Finally, an illustrative numerical experiment is presented in Section 4.

2. A priori error estimates

In this section, the optimality conditions are derived, and the spectral approximation of the control problem is presented. Then a priori error estimates are established. Let $\boldsymbol{U} = L^2(\Omega)^2$, $\boldsymbol{Y} = H_0^1(\Omega)^2$, $Q = L_0^2(\Omega) = \left\{q \in L^2(\Omega) \mid \int_\Omega q dx = 0\right\}$. We consider the