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Fully Discrete Galerkin Finite Element Method for the Cubic Nonlinear Schrödinger Equation

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Abstract. This paper is concerned with numerical method for a two-dimensional time-dependent cubic nonlinear Schrödinger equation. The approximations are obtained by the Galerkin finite element method in space in conjunction with the backward Euler method and the Crank-Nicolson method in time, respectively. We prove optimal L^2 error estimates for two fully discrete schemes by using elliptic projection operator. Finally, a numerical example is provided to verify our theoretical results.

AMS subject classifications: 65M60, 65M50, 65M12

Key words: Finite element method, nonlinear Schrödinger equations, backward Euler scheme, Crank-Nicolson scheme.

1. Introduction

In this paper, we consider the following initial boundary value problem for the two-dimensional time-dependent cubic nonlinear Schrödinger equation: we seek a complex-valued function $u=u(x,y,t)$ defined on $\Omega\times[0,T]$ satisfying

$$\begin{cases} iu_t=-\Delta u+Vu+|u|^2u+f, & (x,y,t)\in\Omega\times[0,T],\\ u(x,y,t)=0, & (x,y,t)\in\partial\Omega\times[0,T],\\ u(x,y,0)=u_0(x,y), & (x,y)\in\Omega, \end{cases} \tag{1.1}$$

where $i=\sqrt{-1}$ is the complex unit, $\Omega\in R^2$ is a convex polygonal domain, Δ is the usual Laplace operator and $u_0(x,y)$ is a given complex-valued initial data. Function $V(x,y)$ is a given real-valued external potential and non-negative bounded for all $(x,y)\in\Omega$. Function $f(x,y,t)$ is complex-valued.

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The Schrödinger equation is the fundamental equation of physics for describing quantum mechanical behavior. Especially, the nonlinear Schrödinger equation (NLSE) can be met in many different areas of physics and chemistry. There are numerous works in the literature to solve the Schrödinger equations. At present, the common numerical methods are finite difference method [1, 7, 10], spectral method [6, 8], two-grid method [12, 29], mixed finite method [19, 30], discontinuous Galerkin method [16, 20, 28] and finite element method [2–5,9,11,13–15,17,18,21–27]. In [1], Akrivis et al. analyzed the discretization of an initial-boundary value problem for the cubic Schrödinger equation in one space dimension by a Crank-Nicolson-type finite difference scheme. In [7], Chang et al. presented several finite difference schemes for the generalized nonlinear Schrödinger equation. Their numerical results demonstrate that the linearized Crank-Nicolson scheme is efficient and robust. In [6], Bao et al. studied the performance of time-splitting spectral approximations for general nonlinear Schrödinger equations in the semiclassical regimes. In [29], Jin et al. solved the time-dependent Schrödinger equation by the finite element two-grid method and analyzed the convergence. The semi-discrete schemes are proved to be convergent with an optimal convergence order and the full-discrete schemes are verified by a numerical example. In [30], Zhao et al. established a new mixed finite element approximate formulation with less degree of freedoms for nonlinear Schrödinger equation based on spaces of bilinear finite element and its gradient. They analyzed error estimates under both semi-discrete and fully-discrete schemes. In [16], Karakashian et al. analyzed the convergence of the discontinuous Galerkin method for the nonlinear (cubic) Schrödinger equation. They showed the existence of the resulting approximations and proved optimal order error estimates in $L^\infty(L^2)$. In [20], Huang et al. analyzed a class of mass preserving direct discontinuous Galerkin (DDG) schemes for Schrödinger equations subject to both linear and nonlinear potentials. In an earlier work [21], Sanz-Serna established Optimal L^2 rates of convergence for several fully-discrete schemes for the numerical solution of the nonlinear Schrödinger equation. In a later work [2], Akrivis et al. approximated the solutions of a nonlinear Schrödinger equations by two fully discrete finite element schemes based on the standard Galerkin method in space and two implicit Crank-Nicolson method in time. The authors studied the existence and uniqueness of solutions and proved L^2 error bounds of optimal order of accuracy with time step condition $\tau = o(h^{\frac{d}{4}}), (d = 2, 3)$. In [24], Tourigny obtained the optimal H^1 error estimates of the backward Euler and Crank-Nicolson Galerkin FEMs for the generalized nonlinear Schrödinger equation with the time step conditions $\tau = o(h^{\frac{d}{2}})$ and $\tau = o(h^{\frac{d}{4}}), (d = 1, 2, 3)$, respectively. In a recent work [26], Wang studied linearized Crank-Nicolson Galerkin FEMs for a generalized nonlinear Schrödinger equation. The author used a complex time-discrete system to present the optimal L^2 error estimate without time-step restrictions. More recently, In [27], Huang et al. considered a two-dimensional time-dependent linear Schrödinger equation with the rectangular Lagrange type finite element of order p, and analyzed the superconvergence error estimate in the semi-discrete scheme and the fully discrete scheme, respectively. In this paper, we consider a cubic nonlinear Schrödinger Eq. (1.1) and analyze the optimal L^2 error estimate in the backward Euler and the Crank-Nicolson fully discrete schemes,