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Support Recovery from Noisy Measurement via Orthogonal Multi-Matching Pursuit

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Abstract. In this paper, a new stopping rule is proposed for orthogonal multi-matching pursuit (OMMP). We show that, for ℓ_2 bounded noise case, OMMP with the new stopping rule can recover the true support of any K-sparse signal x from noisy measurements $y = \Phi x + e$ in at most K iterations, provided that all the nonzero components of x and the elements of the matrix Φ satisfy certain requirements. The proposed method can improve the existing result. In particular, for the noiseless case, OMMP can exactly recover any K-sparse signal under the same RIP condition.

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Key words: sparse recovery, orthogonal matching pursuit, restricted isometry property.

1. Introduction

Sparse recovery problems arise in many applications ranging from signal processing to medical imaging. Suppose x is an unknown n-dimensional signal with at most $K \ll n$ nonzero components:

$$||x||_0 = |supp(x)| = |j : x_j \neq 0| \leq K.$$

We call such signals K-sparse. Orthogonal matching pursuit is a canonical greedy algorithm to approximate the sparse signal from fewer linear measurements

$$y = \Phi x + e,$$

where Φ is an $m \times n (m \ll n)$ sampling matrix, and e is the noise term [5, 6, 8, 11]. Suppose $\Phi = (\Phi_1, \Phi_2, \cdots, \Phi_n)$, where Φ_i denotes the i-th column of Φ. Throughout the paper, we shall assume that the columns of Φ are normalized, i.e., $||\Phi_i||_2 = 1$ for $i = 1, 2, \cdots, n$.

Orthogonal multi-matching pursuit (OMMP), which can be described in Algorithm 1, is a natural generalization of the standard OMP [7, 12, 14]. The key modification

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of OMMP is that it selects N $(N \geq 1)$ indices at each iteration adding to the index set. Comparing with the standard OMP, OMMP recovers the sparse signal in fewer iterations and further decreases the computational complexity.

Algorithm 1.1. Orthogonal multi-matching pursuit - OMMP(Φ, y, N)

Inputs: Sampling matrix Φ, observation y, positive integer N and stopping rule
Outputs: Reconstructed sparse signal x^* and index set
INITIALIZATION: Let the index set $\Omega_0 = \varnothing$ and the residual $r_0 = y$. Let the iteration counter $t = 1$.
IDENTIFICATION: Choose the indices $i_1, i_2, \cdots, i_N$ such that

$$|\Phi_{i_1}^T r_{t-1}| \geq |\Phi_{i_2}^T r_{t-1}| \geq \cdots \geq |\Phi_{i_N}^T r_{t-1}| \geq \max_{j \neq i_1, \cdots, i_N} |\Phi_j^T r_{t-1}|.$$

UPDATE: Add the new indices to the index set: $\Omega_t = \Omega_{t-1} \cup \{i_1, \cdots, i_N\}$, and update the signal and the residual

$$x_t|_{\Omega_t} = \arg\min_z ||y - \Phi_{\Omega_t} z||_2, \quad x_t|_{\Omega_t^c} = 0;$$
$$r_t = y - \Phi x_t.$$

If the stopping rule is achieved, stop the algorithm and the reconstructed sparse signal $x^* = x_t$. Otherwise, update the iteration counter $t = t + 1$ and return to Step IDENTIFICATION.

A widely used condition for sparse recovery is the restricted isometry property (RIP) of the sampling matrix Φ which is introduced by Candes and Tao in [2].

Definition 1.1. *A matrix Φ satisfies the RIP of order K if there exists a constant $\delta_K \in (0, 1)$ such that*

$$(1 - \delta_K)||x||_2^2 \leq ||\Phi x||_2^2 \leq (1 + \delta_K)||x||_2^2 \tag{1.1}$$

holds for all K-sparse signals x. The least number δ_K for which (1.1) *holds is called the restricted isometry constant (RIC).*

Definition 1.2. *The restricted orthogonality constant $\theta_{K,K'}$ is defined as the smallest quantity such that*

$$|\langle \Phi x, \Phi x' \rangle| \leq \theta_{K,K'} ||x||_2 \cdot ||x'||_2 \tag{1.2}$$

holds for all disjoint support K-sparse signals x and K'-sparse signals x'.

For the noiseless case, the theoretical guaranty of OMMP for exact recovery under the RIP of Φ is $\delta_{KN} < \frac{\sqrt{N}}{\sqrt{K} + 3\sqrt{N}}$ [12], $\delta_{KN} < \frac{\sqrt{N}}{\sqrt{K} + 2\sqrt{N}}$ [10]. In [4], the bound was related to

$$\delta_{KN-N+1} + \sqrt{\frac{K}{N}} \theta_{KN-N+1,N} < 1. \tag{1.3}$$