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An Ulm-like Cayley Transform Method for Inverse Eigenvalue Problems with Multiple Eigenvalues

Weiping Shen[1,*], Chong Li[2] and Xiaoqing Jin[3]

[1] *Department of Mathematics, Zhejiang Normal University, Jinhua, 321004, China*
[2] *School of Mathematical Sciences, Zhejiang University, Hangzhou, 310027, China*
[3] *Department of Mathematics, University of Macau, Macao, China.*

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Abstract. We study the convergence of an Ulm-like Cayley transform method for solving inverse eigenvalue problems which avoids solving approximate Jacobian equations. Under the nonsingularity assumption of the relative generalized Jacobian matrices at the solution, a convergence analysis covering both the distinct and multiple eigenvalues cases is provided and the quadratical convergence is proved. Moreover, numerical experiments are given in the last section to illustrate our results.

AMS subject classifications: 65F18, 65F10, 65F15

Key words: Nonlinear equation, inverse eigenvalue problem, Ulm-like method.

1. Introduction

Let $\{A_i\}_{i=0}^n$ be a sequence of real symmetric $n \times n$ matrices. Define the operator $A(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times n}$ by

$$A(\mathbf{c}) := A_0 + \sum_{i=1}^{n} c_i A_i, \quad \text{for any } \mathbf{c} \in \mathbb{R}^n. \tag{1.1}$$

Let $\mathbf{c}= (c_1, c_2, \ldots, c_n)^T \in \mathbb{R}^n$ and denote the eigenvalues of $A(\mathbf{c})$ by $\{\lambda_i(\mathbf{c})\}_{i=1}^n$ with the order $\lambda_1(\mathbf{c}) \leq \lambda_2(\mathbf{c}) \leq \cdots \leq \lambda_n(\mathbf{c})$. The inverse eigenvalue problem (IEP) considered here is, for n given real numbers $\{\lambda_i^*\}_{i=1}^n$ ordering with

$$\lambda_1^* \leq \lambda_2^* \leq \cdots \leq \lambda_n^*, \tag{1.2}$$

*Corresponding author. *Email addresses:* shenweiping@zjnu.cn (W.-P. Shen), cli@zju.edu.cn (C. Li), xqjin@umac.mo (X.-Q. Jin)

to find a vector $\mathbf{c}^* \in \mathbb{R}^n$ such that

$$\lambda_i(\mathbf{c}^*) = \lambda_i^*, \quad \text{for each } i = 1, \ldots, n. \tag{1.3}$$

The vector $\mathbf{c}^*$ is called a solution of the IEP (1.3).

This type of IEP arises in a variety of applications, such as the solution of inverse Sturm-Liouville problems, inverse vibrating string problem, nuclear spectroscopy, molecular spectroscopy, applied mechanics, and so on. For applications, theories and algorithms for the IEP (1.3), one may refer to [1–3, 8, 9, 12, 14, 15, 18, 19, 21, 22, 24, 26–30, 32, 33, 36].

As noted in [1, 4, 30, 36], solving the IEP (1.3) is equivalent to finding a solution $\mathbf{c}^* \in \mathbb{R}^n$ of the nonlinear equation $\mathbf{f}(\mathbf{c}) = \mathbf{0}$, where the function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\mathbf{f}(\mathbf{c}) := (\lambda_1(\mathbf{c}) - \lambda_1^*,\ \lambda_2(\mathbf{c}) - \lambda_2^*,\ \ldots,\ \lambda_n(\mathbf{c}) - \lambda_n^*)^T, \quad \text{for any } \mathbf{c} \in \mathbb{R}^n. \tag{1.4}$$

Then, in the case when the eigenvalues are distinct, i.e.,

$$\lambda_1^* < \lambda_2^* < \cdots < \lambda_n^*, \tag{1.5}$$

Newton's method can be employed to the IEP, and it converges quadratically [4, 12, 20, 36]. However, in the case when multiple eigenvalues are presented (i.e., only (1.2) rather than (1.5) is assumed), the classical convergence analysis for Newton's method does not work because the differentiability of f and the continuity of the eigenvectors corresponding to the multiple eigenvalues fail to hold (cf. [12]). By using the tool of the relative generalized Jacobian of the eigenvalue function (see Proposition 2.1 for the definition), Sun $et\ al.$ proposed in [31] a generalized Newton method for solving the IEP with multiple eigenvalues and established a quadratic convergence under the nonsingularity assumption in terms of the relative generalized Jacobian matrices evaluated at the solution $\mathbf{c}^*$. However, Newton's method (including the generalized Newton method) has the following two disadvantages from the point of view of practical calculation:

(a) it requires solving a complete eigenproblem for the matrix $A(\mathbf{c})$ at each outer iteration;

(b) it requires solving exactly a Jacobian equation at each outer iteration.

This sometimes makes Newton's method inefficient especially when the problem size is large. Therefore, many researchers devoted themselves to overcome these two disadvantages. To avoid the drawback (a), different Newton-type methods for solving the IEP (1.3) have been proposed and studied (cf. [5, 7, 12, 36]). In particular, by applying matrix exponentials and Cayley transforms, a Cayley transform method was designed in [12] and proved to be quadratically convergent both in the distinct and multiple case. To avoid the drawback (b), one approach is to solve the (approximate) Jacobian