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Numerical Approximation to A Stochastic Parabolic PDE with Weak Galerkin Method

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Abstract. The weak Galerkin finite element method is a class of recently and rapidly developed numerical tools for approximating partial differential equations. Unlike the standard Galerkin method, its trial and test function spaces consist of totally discontinuous piecewisely defined polynomials in the whole domain. This method has been vastly applied to many fields [22, 28, 31, 44, 50–52]. In this paper, we will apply this method to approximate a stochastic parabolic partial differential equation. We set up a semi-discrete numerical scheme for the stochastic partial differential equations and derive the optimal order for error estimates in the sense of strong convergence.

AMS subject classifications: 60H15, 65C30, 65M60

Key words: Weak Galerkin method, weak gradient, stochastic PDE.

1. Introduction

There are many issues of uncertainty in applications that the determined equations can not approximate properly [10, 16, 25, 27, 33, 43]. For example, the irregular motion of pollen particles suspended in water, it can be observed that the path of a given particle is very irregular, having a tangent at no point, and the motions of two distinct particles appear to be independent [15]. The stochastic partial differential equations (SPDEs for short) are used to describe a great amount of classical stochastic phenomena in physics [5], chemistry [39], biology [2], etc. The general form of such equations is usually formulated as

$$du = (Au + F(u))dt + B(u)dW(t), \quad u(0) = u_0, \tag{1.1}$$

where H is a Hilbert space, $u(t)$ is an H-valued stochastic process, A is a linear, selfadjoint, positive definite, not necessarily bounded operator with a compact inverse and densely defined in $\mathscr{D} \subset H$. Here, F and B are nonlinear operators on H, $W(t)$ is an H-valued Q-Wiener process defined in a filtered probability space $(\Omega, \mathscr{F}, P, \mathscr{F}_t)$ and $u_0 \in H$ is an initial value.

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Obviously, stochastic models are more complex than deterministic ones [25]. For an example, the solution of an SPDE is not simply a function, but rather a random field or a stochastic process which expresses the implicit variability of the system [6,9,34,40]. These random fields or stochastic processes of solutions usually are lack of regularity, which leads to the difficulty for obtaining a numerical solution with a high order convergence rate. This is the reason that SPDEs are able to more fully capture the behavior of interesting phenomena [2,4,5,39]. This also means that the corresponding numerical analysis of the model will require new tools to model the systems, produce the solutions, and analyze the information stored within the solutions.

In the past decade, there has been tremendous interests and efforts in developing practical and efficient numerical algorithms as well as rigourous error analysis for solving SPDEs [13,17,26,35]. The stochastic problem (1.1) has been treated by various numerical methods which can be mainly divided into two types. One is difference method [8,12,14] and the other is approximated by the use of finite dimensional spaces [1,3,36]. While applying both of these two types of methods to SPDEs, a key difficulty is to find a proper manner to approximate the infinitely dimensional white noise by a finite dimensional version. In [1], piecewise constant functions in both time and space are presented to approximate white noise, while [36] provides spectral method, truncating a part of Fourier series for approaching white noise. Chaos expansion theory is investigated [3] for the numerical approximation of the white noise.

In this paper we will apply the recently developed weak Galerkin (WG for short) finite element method to approximate a linear stochastic partial differential equation

$$\begin{cases} du+Audt=dW & \text{in } \mathscr{D}, \quad 0\le t\le T,\\ u=0 & \text{on } \partial\mathscr{D}, \quad 0\le t\le T,\\ u(0)=u_0 & \text{in } \mathscr{D}. \end{cases} \tag{1.2}$$

This equation is a simplified version of Eq. (1.1).

Yan [48,49] applied the standard finite element method to approach the solution process of (1.2) with $A=-\Delta$. He obtained optimal error estimates in L^2-norm and in H^{-1}-norm, respectively. Larsson etc. [19,20] used a mixed finite element method to approximate equation (1.2) with $A=-\Delta^2$, they proved the error estimate is h^β ($\beta>1$) order, where β represents the smoothness of solution process. Cao etc. [37] employed the Argyris finite elements to equation (1.2) with $A=-\Delta^2$ and improve the optimal order h^β with $\beta>0$. They also obtained $h^{1+\beta}$ order in H^{-1}-norm.

Usually, the solution process of Eq. (1.2) is in low regularity. In this paper, we will use the WG method to approach Eq. (1.2). A typical feature of the WG method is that the chosen finite elements are piecewise defined and are totally discontinuous. Comparing this discontinuity of WG method with the low regularity of solution of (1.2), we believe it will work well when we apply the WG elements approximate those functions with low regularity.

The WG method is firstly proposed by Wang and Ye [22,28,29,31,44,45] to solve the second order elliptic problems. Later on, people show great interests about this method