J. Math. Study
doi: 10.4208/jms.v51n2.18.04

Vol. **51**, No. 2, pp. 177-195
June 2018

Generalized Hermite Spectral Method for Nonlinear Fokker-Planck Equations on the Whole Line

Guo Chai and Tian-jun Wang*

Henan University of Science and Technology, Luoyang 471003, P. R. China.

Received July 13, 2017; Accepted February 8, 2018

Abstract. In this paper, we develop a spectral method for the nonlinear Fokker-Planck equations modeling the relaxation of fermion and boson gases. A full-discrete generalized Hermite spectral scheme is constructed. Its convergence and stability are proved strictly. Numerical results show the efficiency of this approach and coincide well with theoretical analysis.

AMS subject classifications: 65M99, 41A30, 35K55

Key words: Nonlinear Fokker-Planck equations, the whole line, generalized Hermite spectral method, full-discrete scheme.

1 Introduction

In recent years, the more and more attentions were paid to the problems defined on unbounded domains. It is natural to take Hermite (or Laguerre) polynomial/functions as basis functions for the problems defined on the whole line (or half line), see for example [4, 8, 11, 14, 16, 22, 23, 25, 28, 32, 33]. In earlier studies, with respect to the problems defined on the whole line, ones take the Hermite orthogonal polynomials with the weight function $e^{\lambda v^2}$ (λ is a non-zero constant) as the basis functions which lead to non-uniform weights. The non-uniform weights are not natural for certain underlying physical problems, see for example [1–3, 10, 11, 13, 17, 19, 26] and the references therein. In addition, for the problems with variable coefficients or nonlinear wave equations, such as the Fokker-Planck equations in statistic physics or Dirac equation in quantum mechanics, the solutions decay to zero at infinity. The non-uniform weights will destroy the symmetry and positive definiteness of the bilinear operators as well as the conservation properties, and then lead to complications in analysis and implementation. In such cases, we prefer to consider approximations by Hermite functions with weight $\chi(v) \equiv 1$ (see [15, 24, 31, 34]).

*Corresponding author. *Email address:* wangtianjun64@163.com (T. Wang)

In this paper, we consider the Fokker-Planck equations modeling the relaxation of fermion and boson gases. Let v be the velocity of particles, $R=\{v\,|\,-\infty<v<\infty\}$. Denote by $W(v,t)$ the probability density. What's more, $W_0(v)$ represents the initial state. For simplicity, let $\partial_z W=\dfrac{\partial W}{\partial z}$, etc.. We think of the following Cauchy problem (see [28]),

$$\begin{cases} \partial_t W(v,t)=\partial_v(vW(v,t)(1+kW(v,t)))+\partial_v^2 W(v,t), & v\in R,\ 0<t\leq T,\ k=\pm 1,\\ W(v,t)\to 0, & |v|\to\infty,\ 0<t\leq T,\\ W(v,0)=W_0(v), & v\in R. \end{cases} \tag{1.1}$$

The Fokker-Planck equations have been put forward as kinetic models for the relaxation to equilibrium for bosons (k=1) and fermions (k=-1), see [5, 9, 20] and the references therein. These models have been introduced as a simplification with respect to Boltzmann-based models as in [7, 21]. Carrillo, Rosado, and Salvarani [6] indicated that in which way the entropy method applies for quantifying explicitly the exponential decay towards Fermi-Dirac and Bose-Einstein distributions in the one-dimensional case. Also, some authors considered Fokker-Planck equations in an infinite channel or the whole line by using Laguerre functions coupled with domain decomposition (see [18, 27, 28]). The domain decomposition method needs much more basis functions and is extremely complicated to analyze and implement for the problems defined on unbounded domains. In particular, it costs a lot of work to match the numerical solutions on the common boundary of the adjacent subdomains. Thus it is more appropriate to consider approximations by Hermite functions $H_n^\sigma(v)$ (see [31]) directly.

This paper aims at expounding on the generalized Hermite spectral method to the numerical solution of problem (1.1). Firstly, the main difficulty in dealing with (1.1) numerically is caused by the two order difference terms of nonlinear $\partial_v(vW(v,t)(1+kW(v,t)))$. To estimate the errors rooted in the nonlinear terms, we need serval inverse inequalities. Next, we need a few lemmas in time t- direction for numerical analysis of the full-discrete scheme of (1.1). In addition, the fact that the terms $\partial_v^2 W(v,t)$ and $\partial_v W(v,t)$ in (1.1) possess the coefficients 1 and v varying from $-\infty$ to ∞, brings some difficulties in actual computation and numerical analysis (see [28]). To remedy the deficiency, we need a nonstandard projection of generalized Hermite functions (see [29]). Then a generalized Hermite spectral scheme for (1.1) can be constructed, and its convergence and stability can be proved too. Since in many cases, $W(v,t)$ decays very fast as $v\to\infty$, it is better to use the generalized Hermite functions as the basis functions in actual computation (see [24, 31]). They are $L^2(R)$- orthogonal system with regard to weight $\chi(v)\equiv 1$, lead to a much simplified analysis, more precise error estimates and easier algorithms. We also design an efficient algorithm for actual computation, and present some numerical results showing the efficiency of this approach.

This paper is organized as follows. In Section 2, we introduce some results on the generalized Hermite approximation, which play important roles in the numerical analysis of spectral methods for various differential equations in an infinite interval. Then