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An Alternating Direction Method of Multipliers for the Optimization Problem Constrained with a Stationary Maxwell System

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Abstract. This paper mainly focuses on an efficient numerical method for the optimization problem constrained with a stationary Maxwell system. Following the idea of [32], the edge element is applied to approximate the state variable and the control variable, then the continuous optimal control problem is discretized into a finite dimensional one. The novelty of this paper is the approach for solving the discretized system. Based on the separable structure, an alternating direction method of multipliers (ADMM) is proposed. Furthermore, the global convergence analysis is established in the form of the objective function error, which includes the discretization error by the edge element and the iterative error by ADMM. Finally, numerical simulations are presented to demonstrate the efficiency of the proposed algorithm.

AMS subject classifications: 90C30, 90C33, 65K10, 65M60

Key words: Optimal control problem, stationary Maxwell's equations, Nédélec element, ADMM.

1 Introduction

Many complicated problems in engineering, mathematical finance, physics, and life sciences could be modeled by optimization problems with partial differential equations (PDEs) as constraints. The rising number of real world applications demands further developments in numerical schemes for PDE-constrained optimization problems. Thus far, the numerical method based on finite difference methods, finite volume methods, etc., have been developed for the elliptic equation or parabolic equation constrained optimizations and we refer [6, 9–12, 37], and references therein for the rich literature.

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The stationary Maxwell equations play important roles in many modern technologies and applications such as fusion energy, magnetohydrodynamics, electromagnetic induction heating, signal processing, magnetic levitation, and so on. Since the pioneering works of mathematicians and engineers, including Nédélec [22, 23] and Monk [20], the edge element method for solving Maxwell's equations has been widely accepted in the field of mathematical sciences as well as engineering. And for other numerical methods, please see [1, 16, 21, 25, 26, 31, 36] for the details. Recently, the optimal control of Maxwell's equations has attracted researchers' attentions. For the complexity of this kind of problems, the existing researches are mainly based on the degenerate Maxwell's equations [30, 32, 33]. Here, we also focus on the optimization problem constrained with a stationary Maxwell system, and propose an efficient numerical algorithm with the help of the analysis on the edge element in [32] and ADMM in [5].

Generally speaking, there are two strategies for solving the stationary Maxwell constrained optimization problem as well as the traditional PDE-constrained optimization problems: discretize-then-optimize and optimize-then-discretize algorithms [6]. It is well known that the two strategies are equivalent in the sense that the discretised system of the continuous optimality conditions coincides with the optimality conditions for the discretised minimization problem [33], but they have differences in terms of system structure. The latter approach is mainly based on solving a continuous variational system and the aim of the former is to deal with a discretised optimization problem, which could be solved by other optimization algorithms except for the first order optimal condition. Furthermore, based on whether the state variable can be expressed as a function of the control variable directly or not, the numerical methods are divided into reduced space methods and full space methods. Full space methods often are used to solve the optimal control problems with stationary or nonlinear PDE constraints, and reduced space methods are mainly applied to solve the problems with memory requirements or time-dependent problems.

Here, we only mention two representative algorithms for the stationary Maxwell constrained optimization problems. One is the standard gradient descent method, which is a reduced space method with the discretize-then-optimize strategy. The existence, uniqueness, and regularity of the optimal solution for the Maxwell constrained optimization problems with the pointwise state constraints are shown in [32]. Yousept adopted the edge element method to discretize the reduced problem, and solve the discretized system by the gradient descent method. He also presented the error estimates and obtained a reasonable numerical performance. The other one is the adaptive edge element method, which is a full space method with the optimize-then-discretize strategy. Xu and Zou proposed an adaptive edge element method to solve the KKT system associated with the original optimal control problem [30]. A posteriori error estimator of the residue type is derived for the lowest-order edge element approximation. They have also proved that the sequence of discrete solutions converges strongly to the exact solution and the error estimator has a vanishing limit. We refer to [2–4, 24] for more details on the theoretical and computational analysis for the optimization problem constrained by Maxwell's