INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 15, Number 4-5, Pages 479–491

WEAK SOLUTIONS CONSTRUCTED BY SEMI-DISCRETIZATION ARE SUITABLE: THE CASE OF SLIP BOUNDARY CONDITIONS

LUIGI C. BERSELLI

(Communicated by A. Labovsky)

Dedicated to William (Bill) Layton on the occasion of his 60th birthday

Abstract. We consider the initial boundary value problem for the three dimensional Navier-Stokes equations with Navier-type slip boundary conditions. After having properly formulated the problem, we prove that weak solutions constructed by approximating the time-derivative by backward finite differences (with Euler schemes) are suitable. The main novelty is the proof of the local energy inequality in the case of a weak solution constructed by time discretization. Moreover, the problem is analyzed with boundary conditions which are of particular interest in view of applications to turbulent flows.

Key words. Navier-Stokes equations, Euler scheme, local energy inequality, slip boundary conditions.

1. Introduction

In this paper we consider the three dimensional Navier-Stokes equations, with unit viscosity and zero external force (assumptions which are nevertheless unessential) in a bounded domain $\Omega \subset \mathbb{R}^3$, with a smooth boundary $\Gamma = \partial\Omega$ and under "curl based" Navier-type slip boundary conditions. Namely we consider the following initial boundary value problem

$$
\text{(1)} \qquad \begin{cases} \partial_t v - \Delta v + (v \cdot \nabla)\, v + \nabla q = 0 & (t,x) \in]0,T[\times\Omega, \\ \nabla \cdot v = 0 & (t,x) \in]0,T[\times\Omega, \\ v \cdot n = 0 & (t,x) \in]0,T[\times\Gamma, \\ \omega \times n = 0 & (t,x) \in]0,T[\times\Gamma, \\ v(0,x) = v_0(x) & x \in \Omega, \end{cases}
$$

where $v : [0,T] \times \Omega \to \mathbb{R}^3$ is the unknown velocity, $\omega := \operatorname{curl} v$ the vorticity field, and $q : [0,T] \to \mathbb{R}$ the kinematic pressure. The role of the above boundary conditions in the mathematical theory of Euler and Navier-Stokes equations is emphasized in Xiao and Xin [35] and Beirão da Veiga and Crispo [6]. Interesting applications of the above conditions to turbulence modeling, especially for the description of unsteady phenomena, can be found in Layton [26], and the review paper [7] (see also Ref. [10] for a two-dimensional related problem linked with the detection of time-transient phenomena).

In this paper we continue and extend previous work from [13] (done in the space-periodic setting, as well as the forthcoming [9] concerning the space-time discrete problem) and we observe that the analysis of the time-discretization, is a topic which did not attract a lot of attention, in the context of construction of solutions

Received by the editors December 1, 2016 and, in revised form, April 20, 2017.
2000 *Mathematics Subject Classification.* 35Q30, 35A35, 65M20, 76M20.

satisfying the local energy inequality. Starting from the celebrated papers by Scheffer [30] and Caffarelli, Kohn, and Nirenberg [16] concerning the partial regularity for the Navier-Stokes equations, the notion of *suitable weak solution* became a concept of paramount importance in the mathematical theory of the Navier-Stokes equations. We recall that Leray-Hopf weak solutions satisfy a "global energy inequality," while the results of partial regularity require (beside technical conditions on the pressure) the so-called "local energy inequality", see (4) and the next section for precise definitions. In [16] authors introduced an approximation scheme with time-retarded mollifiers, in order to prove the local energy inequality and to estimate the pressure in appropriate Lebesgue spaces. The role of the regularity of the pressure has been later considered in Lin [27] and Vasseur [34]. Combined with the lack of uniqueness of weak solutions, the notion of local energy inequality raised the question to determine which solutions are suitable, see Beirão da Veiga [2, 3, 4] (on the other hand local-in-time strong solutions clearly satisfy the local energy inequality). Especially the question whether or not solutions obtained by the Faedo-Galerkin method satisfy the local energy inequality turned out to be a particular difficult problem. This has been left open for twenty years and a first partial solution to this problem came with the two companion papers by Guermond [22, 23]. In the above references it has been proved that if projectors over the finite element spaces used to discretize (with respect to the space variables x) velocity and pressure satisfy certain *commutation* properties, then weak solutions constructed in the limit of vanishing mesh-size are suitable. In particular, these results cover the MINI element and the Taylor-Hood one. I wrote that this result is partial since –at present– the case of the Fourier-Galerkin method in the space periodic setting is still open, see also Biryuk, Craig, and Ibrahim [15]. The question is also of relevance for applications, because the notion of suitable should be satisfied by any reasonable solution (called "physically relevant") obtained with approximation by Large Eddy Simulation methods, see Guermond *et al.* [24, 25]. Other recent related results can be found in [13, 14, 19].

In this paper, we continue in the spirit of connecting results from mathematical analysis with those from numerical analysis, and we focus on understanding when discrete-time approximations produce suitable solutions, as the time-step-size $\kappa > 0$ goes to zero. We treat the boundary value problem with certain slip conditions, while the Dirichlet problem seems to require a completely different and much more technical treatment, which is object of a still ongoing research. **Note added in proof:** After the paper being accepted we have been aware that in Sec. 5 of Ref [21] the time-discrete problem in the implicit case is studied in the Dirichlet case, by using techniques of semigroup theory.

In particular, we analyze the following single step scheme:

Algorithm. (Euler implicit) Let be given a time-step-size $\kappa > 0$ and the corresponding net $I^M = \{t_m\}_{m=0}^M$, with $M = [T/\kappa] \in \mathbb{N}$ and $t_m := m\kappa$. Then, for $m \geq 1$ and for v^{m-1} given from the previous step with $v^0 = v_0$, compute the iterate v^m as follows: Solve

$$
\begin{cases}
d_t v^m - \Delta v^m + (v^m \cdot \nabla)\, v^m + \nabla q^m = 0 & \text{in } \Omega, \\
\nabla \cdot v^m = 0 & \text{in } \Omega, \\
v^m \cdot n = 0 & \text{on } \Gamma, \\
\operatorname{curl} v^m = 0 & \text{on } \Gamma,
\end{cases}
\tag{2}
$$

where $d_t v^m := \frac{v^m - v^{m-1}}{\kappa}$ denotes the backward finite difference.