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EXISTENCE AND ERGODICITY FOR THE TWO-DIMENSIONAL STOCHASTIC BOUSSINESQ EQUATION

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This paper is dedicated to our friend and mentor William Layton.

Abstract. The existence of solutions to the Boussinesq system driven by random exterior forcing terms both in the velocity field and the temperature is proven using a semigroup approach. We also obtain the existence and uniqueness of an invariant measure via coupling methods.

Key words. Stochastic Boussinesq equation, invariant measure, coupling, ergodicity.

1. Introduction

We study the existence and ergodicity of the stochastic Boussinesq equation

$$\begin{aligned}
&du = (\nu\Delta u - (u\cdot\nabla)u + \sigma\theta - \nabla p)dt + \sqrt{Q_1}dW_1(t),\\
&d\theta = (\chi\Delta\theta - (u\cdot\nabla)\theta)dt + \sqrt{Q_2}dW_2(t),\\
&\nabla\cdot u = 0 \quad \text{in } (0,+\infty)\times\mathcal{O},\\
&u = 0, \quad \theta = 0 \quad \text{on } (0,+\infty)\times\partial\mathcal{O},\\
&u(0,x) = u_0(x), \quad \theta(0,x) = \theta_0(x) \quad \text{in } \mathcal{O},
\end{aligned} \tag{1}$$

which models the interactions between an incompressible fluid flow coupled with thermal dynamics in two dimensions, in the presence of random perturbations. Here $\mathcal{O}\subset\mathbb{R}^2$ is a bounded, open and simply connected domain with smooth boundary $\partial\mathcal{O}$, and $u = (u_1, u_2)$ denotes the fluid velocity field, θ is the temperature of the fluid, p stands for the pressure, ν is the kinematic viscosity and χ is the thermal diffusivity, σ is a constant two component-vector. Also W_1 and W_2 represent two independent cylindrical Wiener processes [18, 21] defined, respectively, on a filtered space $(\Omega, \mathcal{F}_t, \mathbb{P})$ taking values in

$$H = \left\{ v\in \left(L^2(\mathcal{O})\right)^2 : \nabla\cdot v = 0 \text{ in } \mathcal{O}, \quad v\cdot n = 0 \text{ on } \partial\mathcal{O} \right\}, \quad H_1 = L^2(\mathcal{O}).$$

Finally, Q_1 and Q_2 are linear continuous, positive and symmetric operators on H and H_1, respectively, of trace class (see Definition A.1 in the Appendix A), i.e., $Tr\,Q_i < \infty$, $i = 1, 2$, satisfying the following condition:

$$Q_1 = A^{-\gamma}, \qquad Q_2 = A_1^{-\gamma}, \tag{2}$$

where $1/2 < \gamma < 1$, A and A_1 are as defined in (3).

Herein we prove the existence and uniqueness of a solution $(u(t, u_0, \theta_0), \theta(t, u_0, \theta_0))$ of the stochastic Boussinesq system (1), and of the corresponding invariant measure in the space $H \times H_1$. The deterministic version of the Boussinesq system (1) was comprehensively studied in the literature (see, e.g. [1, 16, 25] and the references therein). In the 19th century, Boussinesq conjectured that turbulent flow cannot be described solely with deterministic methods, and indicated that a

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stochastic framework should be used [23]. In the case of two-dimensional Navier-Stokes equations, the existence and uniqueness of a solution, the uniqueness of the invariant measure and properties of the corresponding Kolmogorov operators were studied in [3, 7, 6, 12, 11]. For the two-dimensional magnetohydrodynamics system, see [2, 24, 5, 22, 15, 14]. Recently, many authors have studied ergodicity for the solutions of the stochastic Boussinesq equations [10, 26, 27, 4, 9, 13, 17]. Notwithstanding the physical differences between the Navier-Stokes equations, magnetohydrodynamics and the Boussinesq equations (different conserved quantities, unitless physical parameters, energy cascades, fine scale structure of flows, complex pattern formation, mean heat transport, Alfvén waves, wavepackets), from a functional analysis viewpoint, similar results hold. There is an increasing set of standard tools which can be often adapted to prove deterministic and statistical properties for all these flows. Inhere we follow closely [2] by adjusting most of the proofs to the fact that the temperature field is non-solenoidal.

The paper is organized as follows. In Section 2 we formulate problem (1) in an appropriate functional setting (see [25, 8, 21, 18]) and in Section 3 we give the main existence and uniqueness result for (1) which is proved via an approximating regularizing scheme. In Section 4 we prove the existence of an invariant measure μ corresponding to the stochastic flow $t \mapsto (u(t), \theta(t))$, and its uniqueness via coupling methods, following [19, 2]. In particular, the uniqueness of the invariant measure implies that the flow is ergodic, i.e.,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \phi(u(t), \theta(t))\, dt = \int_{H\times H} \phi\, d\mu \qquad \forall \phi \in L^2(H \times H; \mu)$$

which agrees with some physical hypothesis on the Boussinesq flow. In the concluding Section 5 we summarize our results, in the Appendix A we provide some definitions used throughout the report, while Section 6 contains acknowledgements.

2. Functional setting and formulation of the problem

We introduce the functional spaces to represent the coupled Navier-Stokes and heat equations (1) as infinite dimensional differential equation

$$V = \left\{ v \in \left(H_0^1(\mathcal{O})\right)^2 : \nabla \cdot v = 0 \text{ in } \mathcal{O} \right\}, \quad V_1 = H_0^1(\mathcal{O}).$$

The norms of V and V_1 are denoted by the same symbol $\|\cdot\|$:

$$\|v\|^2 = \sum_{i=1}^{2} \int_{\mathcal{O}} |\nabla v_i|^2 dx, \quad v = (v_1, v_2) \in V,$$
$$\|\eta\|^2 = \int_{\mathcal{O}} |\nabla \eta|^2 dx, \quad \eta \in V_1.$$

Let denote by V' and $V_1' = H^{-1}(\mathcal{O})$ the duals of V and V_1 respectively, endowed with the dual norms. Denote again $(\cdot, \cdot)$ the scalar product on H and the pairing between V and V', V_1 and V_1'. The norm on H and $L^2(\mathcal{O})$ will both be denoted by $|\cdot|$. Identifying H with its own dual we have $V \subset H \subset V'$. Let $A \in L(V, V')$, $A_1 \in$