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Some Generalized q-Bessel Type Wavelets and Associated Transforms

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Abstract. In this paper wavelet functions are introduced in the context of q-theory. We precisely extend the case of Bessel and q-Bessel wavelets to the generalized q-Bessel wavelets. Starting from the (q,v)-extension ($v=(\alpha,\beta)$) of the q-case, associated generalized q-wavelets and generalized q-wavelet transforms are developed for the new context. Reconstruction and Placherel type formulas are proved.

Key Words: Wavelets, Besel function, q-Bessel function, modified Bessel functions, generalized q-Bessel functions, q-Bessel wavelets.

AMS Subject Classifications: 42A38, 42C40, 33D05, 26D15

1 Introduction and brief review

Wavelet theory has been known a great success since the eighteenth of the last century. It provides for function spaces as well as time series good bases allowing the decomposition of the studied object into spices associated to different horizons known as the levels of decomposition. A wavelet basis is a family of functions obtained from one function known as the mother wavelet, by translations and dilations. Due to the power of their theory, wavelets have many applications in different domains such as mathematics, physics, electrical engineering, seismic geology. This tool permits the representation of L^2-functions in a basis well localized in time and in frequency. Hence, wavelets are special functions characterized by special properties that may not be satisfied by other functions. In the present context, our aim is to develop new wavelet functions based on

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some special functions such as Bessel one. Bessel functions form an important class of special functions and are applied almost everywhere in mathematical physics. They are also known as cylindrical functions, or cylindrical harmonics, because of their strong link to the solutions of the Laplace equation in cylindrical coordinates. We aim precisely to apply the generalized q-Bessel function introduced in the context of q-theory and which makes a general variant of Bessel, Bessel modified and q-Bessel functions.

To organize this paper, we will briefly review in the rest of this section the wavelet theory on the real line. In Section 2, the basic concepts on Bessel wavelets are presented. Section 3 is devoted to the presentation of the extension to the q-Bessel wavelets. Section 4 is concerned with the developments of our new extension to the case of generalized q-Bessel wavelets. Backgrounds on wavelets, q-theory, q-wavelets and related topics may be found in [2,5,6,8,10,11,26] and the references therein.

We now recall some basic definitions and properties of wavelets on $\mathbb{R}$. For more details we may refer to [18,24]. In $L^2(\mathbb{R})$, a wavelet is a function $\psi\in L^2(\mathbb{R})$ satisfying the so-called admissibility condition

$$C_\psi=\int_0^\infty \frac{|\widehat{\psi}(\xi)|^2}{\xi}d\xi<\infty.$$

From translations and dilations of ψ, we obtain a family of wavelets $\{\psi_{a,b}\}$

$$\psi_{a,b}(x)=\frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right),\quad b\in\mathbb{R},\quad a>0. \tag{1.1}$$

ψ is called the mother wavelet. **a** is the parameter of dilation (or scale) and **b** is the parameter of translation (or position).

The continuous wavelet transform of a function $f\in L^2(\mathbb{R})$ at the scale a and the position b is given by

$$C_f(a,b)=\int_{-\infty}^{+\infty} f(t)\overline{\psi}_{a,b}(t)dt.$$

The wavelet transform $C_f(a,b)$ has several properties.

- It is linear, in the sense that

$$C_{(\alpha f_1+\beta f_2)}(a,b)=\alpha C_{f_1}(a,b)+\beta C_{f_2}(a,b),\quad \forall\alpha,\beta\in\mathbb{R}\quad\text{and}\quad f_1,f_2\in L^2(\mathbb{R}).$$

- It is translation invariant:

$$C_{(\tau_{b'}f)}(a,b)=C_f(a,b-b'),$$

where $\tau_{b'}$ refers to the translation of f by b' given by

$$(\tau_{b'}f)(x)=f(x-b').$$