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# Boundedness Estimates for Commutators of Riesz Transforms Related to Schrödinger Operators

Yueshan Wang* and Yuexiang He

*Department of Mathematics, Jiaozuo University, Jiaozuo 454003, Henan, China*

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**Abstract.** Let $\mathcal{L}=-\Delta+V$ be a Schrödinger operator on $\mathbb{R}^n(n\geq 3)$, where the nonnegative potential $V$ belongs to reverse Hölder class $RH_{q_1}$ for $q_1>\frac{n}{2}$. Let $H^p_{\mathcal{L}}(\mathbb{R}^n)$ be the Hardy space associated with $\mathcal{L}$. In this paper, we consider the commutator $[b,T_\alpha]$, which associated with the Riesz transform $T_\alpha=V^\alpha(-\Delta+V)^{-\alpha}$ with $0<\alpha\leq 1$, and a locally integrable function $b$ belongs to the new Campanato space $\Lambda^\theta_\beta(\rho)$. We establish the boundedness of $[b,T_\alpha]$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1<p<q_1/\alpha$ with $1/q=1/p-\beta/n$. We also show that $[b,T_\alpha]$ is bounded from $H^p_{\mathcal{L}}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ when $n/(n+\beta)<p\leq 1,1/q=1/p-\beta/n$. Moreover, we prove that $[b,T_\alpha]$ maps $H^{\frac{n}{n+\beta}}_{\mathcal{L}}(\mathbb{R}^n)$ continuously into weak $L^1(\mathbb{R}^n)$.

**Key Words**: Riesz transform, Schrödinger operator, commutator, Campanato space, Hardy space.
**AMS Subject Classifications**: 42B30, 42B25, 35J10

## 1 Introduction and results

Let $\mathcal{L}=-\Delta+V$ be a Schrödinger operator on $\mathbb{R}^n$, where $n\geq 3$. The function $V$ is nonnegative, $V\neq 0$, and belongs to a reverse Hölder class $RH_{q_1}$ for some $q_1>n/2$, that is to say, $V$ satisfies the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(y)^{q_1}dy\right)^{1/q_1}\leq\frac{C}{|B|}\int_B V(y)dy$$

for all ball $B\subset\mathbb{R}^n$. We consider the Riesz transform $T_\alpha=V^\alpha(-\Delta+V)^{-\alpha}$, where $0<\alpha\leq 1$.

Many results about $T_\alpha=V^\alpha(-\Delta+V)^{-\alpha}$ and its commutator have been obtained. Shen [1] established the $L^p$- boundedness of $T_1$ and $T_{1/2}$, Liu and Tang [2] showed that $T_1$ and $T_{1/2}$ are bounded on $H^p_{\mathcal{L}}(\mathbb{R}^n)$ for $\frac{n}{n+\delta'}<p\leq 1$. For $0<\alpha\leq 1$, Sugano [3] studied the $L^p$- boundedness and Hu and Wang [4] obtained the $H^p_{\mathcal{L}}(\mathbb{R}^n)$ boundedness. When $b\in BMO$, Guo,

*Corresponding author. *Email addresses:* `wangys1962@163.com` (Y. S. Wang), `heyuexiang63@163.com` (Y. X. He)

 

Li and Peng [5] obtained the $L^p$-boundedness of commutators $[b,T_1]$ and $[b,T_{1/2}]$, Li and Peng in [6] proved that $[b,T_1]$ and $[b,T_{1/2}]$ map continuously $H^1_{\mathcal{L}}(\mathbb{R}^n)$ into weak $L^1(\mathbb{R}^n)$. When $b\in BMO_\theta(\rho)$ and $0<\alpha\le 1$, the $L^p$- boundedness of $[b,T_\alpha]$ was investigated in [7] and the boundedness from $H^1_{\mathcal{L}}(\mathbb{R}^n)$ into weak $L^1(\mathbb{R}^n)$ given in [4].

In this paper, we are interested in the boundedness of $[b,T_\alpha]$ when $b$ belongs to the new Campanato class $\Lambda^\theta_\beta(\rho)$. Let us recall some concepts.

As in [1], for a given potential $V\in RH_{q_1}$ with $q_1>n/2$, we define the auxiliary function

$$\rho(x)=\sup\left\{r>0:\frac{1}{r^{n-2}}\int_{B(x,r)}V(y)dy\le 1\right\},\quad x\in\mathbb{R}^n.$$

It is well known that $0<\rho(x)<\infty$ for any $x\in\mathbb{R}^n$.

Let $\theta>0$ and $0<\beta<1$, in view of [8], the new Campanato class $\Lambda^\theta_\beta(\rho)$ consists of the locally integrable functions $b$ such that

$$\frac{1}{|B(x,r)|^{1+\beta/n}}\int_{B(x,r)}|b(y)-b_B|dy\le C\left(1+\frac{r}{\rho(x)}\right)^\theta$$

for all $x\in\mathbb{R}^n$ and $r>0$. A seminorm of $b\in\Lambda^\theta_\beta(\rho)$, denoted by $[b]^\theta_\beta$, is given by the infimum of the constants in the inequalities above.

Note that if $\theta=0$, $\Lambda^\theta_\beta(\rho)$ is the classical Campanato space; If $\beta=0,\Lambda^\theta_\beta(\rho)$ is exactly the space $BMO_\theta(\rho)$ introduced in [9].

We recall the Hardy space associated with Schrödinger operator $\mathcal{L}$, which had been studied by Dziubański and Zienkiewicz in [10,11]. Because $V\in L^{q_1}_{loc}(\mathbb{R}^n)$, the Schrödinger operator $\mathcal{L}$ generates a $(C_0)$ contraction semigroup $\{T^{\mathcal{L}}_s:s>0\}=\{e^{-s\mathcal{L}}:s>0\}$. The maximal function associated with $\{T^{\mathcal{L}}_s:s>0\}$ is defined by $M^{\mathcal{L}}f(x)=\sup_{s>0}|T^{\mathcal{L}}_s f(x)|$. we always denote $\delta'=\min\{1,2-n/q_1\}$. For $\frac{n}{n+\delta'}<p\le 1$, We say that $f$ is an element of $H^p_{\mathcal{L}}(\mathbb{R}^n)$ if the maximal function $M^{\mathcal{L}}f$ belongs to $L^p(\mathbb{R}^n)$. The quasi-norm of $f$ is defined by $\|f\|_{H^p_{\mathcal{L}}(\mathbb{R}^n)}=\|M^{\mathcal{L}}f\|_{L^p(\mathbb{R}^n)}$.

We now formulate our main results as follows.

**Theorem 1.1.** *Let $V\in RH_{q_1}$ with $q_1>n/2$, and let $b\in\Lambda^\theta_\beta(\rho)$. If $0<\alpha\le 1$ and $\frac{q_1}{q_1-\alpha}<p<\infty$, then*

$$\|[b,T^*_\alpha]\|_{L^q(\mathbb{R}^n)}\le C[b]^\theta_\beta\|f\|_{L^p(\mathbb{R}^n)},$$

*where $1/q=1/p-\beta/n$, and $T^*_\alpha=(-\Delta+V)^{-\alpha}V^\alpha$.*

We immediately deduce the following result by duality.

**Corollary 1.1.** Let $V\in RH_{q_1}$ with $q_1>n/2$, and let $b\in\Lambda^\theta_\beta(\rho)$. If $0<\alpha\le 1$ and $1<p<q_1/\alpha$, then

$$\|[b,T_\alpha]\|_{L^q(\mathbb{R}^n)}\le C[b]^\theta_\beta\|f\|_{L^p(\mathbb{R}^n)},$$

where $1/q=1/p-\beta/n$.