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# Numerical Methods for System Parabolic Variational Inequalities from Regime-Switching American Option Pricing

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**Abstract.** The aim of this paper is to study the convergence rates of the trinomial tree methods (TTMs) and perturbed finite difference methods (PFDMs) for system parabolic variational inequalities which govern the value function of regime-switching American option. This paper has threefold contributions: (i) It establishes the higher-order equivalence between the TTMs and the PFDMs for the regime-switching American options; (ii) It proves the regularities of the solutions to the system of parabolic variational inequalities governing the price of the American options, and studies the comparison principles and the penalty methods. These results are used to prove the convergence rates of the PFDMs; (iii) It proves the convergence rates of the PFDMs for the system of parabolic variational inequalities governing the price of the American options. The convergence rates of the TTMs are obtained by the higher-order equivalence between the TTMs and the PFDMs and the convergence theory for the PFDMs.

**AMS subject classifications**: 65C20, 65C40, 65M06, 91G20, 91G60

**Key words**: Parabolic variational inequality, American option pricing, trinomial tree methods, finite difference methods, regime-switching models.

## 1. Introduction

Assume that the underlying asset price $S_\tau$ follows a $\varrho$-states regime switching model under risk-neutral measure:

$$\frac{dS(\tau)}{S(\tau)} = r(\alpha(\tau))d\tau + \sigma(\alpha(\tau))dW(\tau), \tag{1.1}$$

where $W(\tau)$ is a standard Brownian motion, $\alpha(\tau)$ is a continuous-time Markov chain with $\varrho$ states $(\alpha_1,\cdots,\alpha_\varrho)$. Assume also that at each state $\alpha(\tau)=\alpha_i$, $i\in\mathbb{D}=\{1,\cdots,\varrho\}$, the interest rate $r(\alpha_i)=r_i\geq 0$ and volatility $\sigma(\alpha_i)=\sigma_i$ for $i\in\mathbb{D}$ is constant. Let $A=(a_{il})_{i,l\in\mathbb{D}}$

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be the generator matrix of the Markov chain process whose elements are constants satisfying $a_{il} \geq 0$ for $i \neq l$ and $\sum_{l=1}^{\varrho} a_{il} = 0$ for $i \in \mathbb{D}$.

The regime-switching model is popular in financial data modelling and analysis as it allows parameters of asset price dynamics to depend on a finite state Markov chain process. It has been shown in the literature that the Markov chain process is effective in providing information of the market environment. There are many empirical studies on the Markov regime switching models (see, e.g., [1,2,4,14] and the references therein).

The aim of this paper is to study the convergence rates of the trinomial tree methods (TTMs) and perturbed finite difference methods (PFDMs) for the American options with the underlying asset price following regime-switching model (1.1). The value of American option under the regime-switching model (1.1) with maturity date $T$ and payoff $\widetilde{g}(S) = (K-S)^+$ satisfies the following system of variational inequalities (see [17]):

$$\min\left\{-\mathscr{L}_i V(S,\tau,i) + \sum_{l=1}^{\varrho} a_{il} V(S,\tau,l), V(S,\tau,i) - \widetilde{g}(S)\right\} = 0, \quad i \in \mathbb{D}, \tag{1.2a}$$

$$V(S,T,i) = \widetilde{g}(S), \qquad i \in \mathbb{D}, \tag{1.2b}$$

where

$$\mathscr{L}_i V(S,\tau,i) = \frac{\partial}{\partial \tau} V(S,\tau,i) + \frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2}{\partial S^2} V(S,\tau,i) + r_i S \frac{\partial}{\partial S} V(S,\tau,i) - r_i V(S,\tau,i).$$

The valuation of options with the price of underlying asset following the regime-switching models has been active for many years. Particularly the trinomial tree methods are developed for the option pricing with regime-switching. Liu [9,10] develops a linear tree for a regime-switching geometric Brownian motion model and extends it to a class of regime-switching mean-reverting models. Yuen and Yang [19] construct efficient trinomial tree methods for European and American option pricing in Markov regime-switching models. Ma and Zhu [11] prove the convergence rates of the trinomial tree of Yuen and Yang [19] for European options. Jiang et al. [6] develop a recombining trinomial tree method for option pricing with state-dependent switching rates. Ma, Tang and Zhu [12] establish the high-order equivalence between the finite difference methods and the trinomial tree methods of [19] for pricing European options with regime-switching models.

In this paper, we construct a PFDM for solving the parabolic variational inequalities (1.2a) and prove that it is equivalent to the TTMs of Yuen and Yang [19] in the higher-order sense. We prove the regularities of the solutions to the system of parabolic variational inequalities governing the price of the American options, the comparison principles and the penalty methods, which are required to establish the theory of convergence rates of the PFDMs. We then prove the convergence rates of the PFDMs for the system of parabolic variational inequalities (1.2a). The convergence rates of the TTMs are obtained by the convergence rate theory for the PFDMs and the higher-order equivalence between the TTMs and the PFDMs. The analysis of this paper covers the no regime-switching case when the parameters of the interest rates and volatilities do not change with the regime-switching. Liang et al. [7] proves the convergence rates of the binomial tree methods