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The Weak Galerkin Method for Elliptic Eigenvalue Problems

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Abstract. This article is devoted to studying the application of the weak Galerkin (WG) finite element method to the elliptic eigenvalue problem with an emphasis on obtaining lower bounds. The WG method uses discontinuous polynomials on polygonal or polyhedral finite element partitions. The non-conforming finite element space of the WG method is the key of the lower bound property. It also makes the WG method more robust and flexible in solving eigenvalue problems. We demonstrate that the WG method can achieve arbitrary high convergence order. This is in contrast with existing nonconforming finite element methods which can provide lower bound approximations by linear finite elements. Numerical results are presented to demonstrate the efficiency and accuracy of the theoretical results.

AMS subject classifications: 65N30, 65N15, 65N12, 74N20, 35B45, 35J50, 35J35

Key words: Weak Galerkin finite element method, elliptic eigenvalue problem, lower bounds, error estimate.

1 Introduction

The eigenvalue problems of partial differential equations arising from the scientific research and engineering have received more and more attentions recently [6, 9, 14, 28]. Among the PDE eigenvalue problems, the elliptic eigenvalues are closely related to Poincaré constant in the Sobolev theory [19, 34], and play an important role in the spectral distribution of nonlinear equations [33]. In physical applications, the eigenvalues

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often have close relationship with vibrations, especially the sympathetic vibration phenomenon. Most elastic bodies vibrate at certain frequency and respond to external vibrations. Details of these applications can be found in [7, 15, 20]. In addition to the above mentioned applications, the elliptic type eigenvalue problems are also useful in many other areas, such as plasma physics in fusion experiments and astrophysics, the petroleum reservoir simulation, the linear stability of flows in fluid mechanics, and electronic band structure calculations, etc. (see [1, 4, 14, 21] and references therein).

There have been numerous efforts in finding numerical solutions of elliptic eigenvalue problems. The finite element method is the most studied method (see, e.g., [3, 4, 6, 16, 17]). Due to the Rayleigh quotient and the minimum-maximum principle, any standard conforming finite element method [13, 35] can only provide upper bounds for the eigenvalues. However, when the eigenvalues are all real numbers, it is desirable to obtain both upper bounds and lower bounds [27].

There are mainly two approaches to find lower bounds of eigenvalues. The first approach is a post-processing procedure [22,26]. The main drawback of this approach is that an auxiliary problem must be solved and the order of convergence is reduced as a result. The second approach is the construction of special nonconforming finite elements to obtain lower bounds. In [23], three types of nonconforming elements by the finite element error expansion technique were studied to provide lower bounds of the eigenvalues. We refer interested readers to [5, 25] for additional studies on other nonconforming finite element approaches in obtaining lower bounds for eigenvalues. There are mainly two difficulties in the numerical approximation of eigenvalue problems with nonconforming FEM. One is the construction of high order finite element spaces which give high order numerical solutions. The other difficulty is the construction of finite element spaces for three dimensional problems.

The goal of this paper is to overcome the aforementioned difficulties using the weak Galerkin method. The WG method was first proposed in [37], and further developed in [8, 29, 31, 36, 38, 44, 45, 47, 48]. Recently, the weak Galerkin method has been extended to elliptic interface problems [10], linear hyperbolic equations [43], Navier-Stokes equations [46, 49], Helmholtz equations [11, 32], and discrete maximum principles [18, 39]. In the WG method, differential operators are approximated by weak forms as distributions over a set of generalized functions. It has been demonstrated that the WG method is highly flexible and robust as a numerical technique employing discontinuous piecewise polynomials on polygonal or polyhedral finite element partitions. As a class of nonconforming finite element method, the finite element space of WG, for the same degree of polynomial, is larger than that of the standard finite element methods, which makes it possible to obtain lower bounds due to the Rayleigh quotient. Comparing with other nonconforming finite element for eigenvalue problems, our approach of solving elliptic eigenvalue problems with the WG method has the following advantages, which constitute the main contributions of this paper.

- Our method is capable of obtaining lower bounds with higher order of accuracy.