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Some Notes on *k*-minimality

Azam Etemad Dehkordy*

Department of Mathematical Sciences, Isfahan University of Technology, Isfahan, Iran.

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Abstract. The concept of minimality is generalized in different ways, one of which is the definition of k-minimality. In this paper k-minimality is studied for minimal hypersurfaces of a Euclidean space under different conditions on the number of principal curvatures. We will also give a counterexample to L_k-conjecture.

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1 Introduction

Let $x: M \to \mathbb{E}^m$ be an isometric immersion from a Riemannian $n-$manifold into a Euclidean space. Denote the Laplacian, the position vector and the mean curvature vector field of M, respectively, by Δ, x and $\vec{H}$. Then, M is called a biharmonic submanifold if $\Delta\vec{H} = 0$. Beltrami's formula, $\Delta x = -n\vec{H}$, implies that every minimal submanifold of $\mathbb{E}^m$ is a biharmonic submanifold.

Chen initiated the study of biharmonic submanifolds in the mid 1980s [4]. Then, Chen and other authors proved that, in specific cases, a biharmonic submanifold is a minimal submanifold [4,5,7] and Chen introduced his famous conjecture [3]. This conjecture remains open, although the study thereof is active nowadays. Among other results, it is proved in [6] that Chen's Conjecture is true for biharmonic hypersurfaces with three distinct principal curvatures in $\mathbb{E}^m$. Furthermore, under a generic condition, Koiso and Urakawa [8] gave affirmative answer to Chen conjecture.

The linearized operator of $(k+1)$-th mean curvature of a hypersurface, i.e. H_{k+1}, is the $\mathrm{L_k}$ operator. The L_k operator is a natural generalization of Laplace operator for $k=1,...,n$ [9,10]. Let $x: M^n \to \mathbb{E}^{n+1}$ be an isometric immersion from a connected orientable Riemannian hypersurface into the Euclidean space $\mathbb{E}^{n+1}$. It is proved that [1]

$$L_k x = (k+1)\begin{pmatrix} n \\ k+1 \end{pmatrix} H_{k+1} N,$$

*Corresponding author. *Email address:* `ae110mat@cc.iut.ac.ir` (A. Etemad)

where N is the unit normal vector field and $k=0,..,n-1$.The L_k-conjecture is as follows.

L_k-Conjecture. Every L_k-biharmonic hypersurface, namely a Euclidean hypersurface $x: M^n \to \mathbb{E}^{n+1}$ satisfying the condition $L_k^2 x = 0$ for some $k=0,...,n-1$, has zero $(k+1)$-th mean curvature.

A manifold with zero $(k+1)$-th mean curvature is called k-minimal for $k=0,..,n-1$. In 2015, Aminian and Kashani [2] proved the L_k-conjecture for Euclidean hypersurfaces with at most two principal curvatures. They also proved the L_k-conjecture for L_k-finite type hypersurfaces.

In this paper, we prove that the L_1-conjecture is not true for a connected minimal hypersurface of a Eucldean space with arbitrary number of principal curvatures.

2 Preliminaries

In this section, we recall some standard definitions and results from Riemannian geometry. Let $n \geq 2$ and suppose $x: M^n \to \mathbb{E}^{n+1}$ is an isometric immersion from an n-dimensional connected Riemannian manifold M^n into Euclidean space $\mathbb{E}^{n+1}$.

Let A be the shape operator of this immersion and $\lambda_1,...,\lambda_n$ be the eigenvalues of this self-adjoint operator. The mean curvature of M is given by

$$nH = \text{trace } A = \lambda_1. \ ... \ .\lambda_n.$$

The $k-$th mean curvature of M is also defined by

$$\binom{n}{k} H_k = s_k,$$

where $s_0 = 1$ and $s_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1}. \ \dots \ .\lambda_{i_k}$, for $k=1,...,n$. It is obvious that $H_1 = H$ and $S = n(n-1)H_2$, where S is the scalar curvature of M.

The Newton transformations $P_k : C^\infty(TM^n) \to C^\infty(TM^n)$ are defined inductively by $P_0 = I$ and

$$P_k = s_k I - A \circ P_{k-1}, \ \ 1 \leq k \leq n.$$

Therefore,

$$P_k = \sum_{i=0}^{k} (-1)^i s_{k-i} A^i, \ \ 1 \leq k \leq n.$$

Thus the Cayley-Hamilton theorem implies that $P_n = 0$. It is well known that each P_k is a self-adjoint linear operator which commutes with A. For $k=0$, ..., n, the second