East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.250518.041118

Vol. **9**, No. 2, pp. 369-385
May 2019

A Fourier Method to Recover Elastic Sources with Multi-Frequency Data

Minghui Song* and Xianchao Wang

Department of Mathematics, Harbin Institute of Technology,
Harbin 150001, P.R. China.

Received 25 May 2018; Accepted (in revised version) 4 November 2018.

Abstract. An inverse source problem for elastic waves in isotropic homogeneous media is considered and a Fourier transform based multi-frequency approach to determine the elastic source is proposed. This is an easily implementable fast non-iterative inversion scheme. The stability estimate is derived. Numerical examples demonstrate the effectiveness of the method.

AMS subject classifications: 49N45, 65L32, 65N21

Key words: Inverse source problem, elastic wave, Fourier transform, multi-frequency, far field.

1. Introduction

Inverse source problems are associated with the reconstruction of an unknown source from the measurement of the radiating field data. They occur in various practical applications, including acoustic tomography [3, 7, 13, 15], medical imaging [2, 4, 10], pollution of environment [8], gesture-computing technology [11, 14, 16] and have been vigorously studied in recent years. In the fixed frequency case, the inverse source problems possess non-unique solutions due to the presence of a non-radiating source [1,6,12,17]. One of the ways to overcome this difficulty consists in the use of multi-frequency data [5]. To identify acoustic and electromagnetic sources, an eigenfunction expansion method was developed in [9, 18] and to approximate such sources with near field data in the frequency domain, the Fourier expansion was used [19, 22]. In [23], multiple multipolar sources are located by a sampling method.

A novel method for the reconstruction of acoustic and electromagnetic sources with far field data has been proposed in our work [20, 21]. The method is based on a multi-frequency Fourier expansion. Here, we extend that approach to inverse source problems for elastic waves. Such an extension is non-trivial because of the presence of different frequencies for compressional and shear waves. In addition, the Navier equation is difficult to analyse and its Green function has a higher-order singularity than the Helmholtz equation

*Corresponding author. *Email addresses:* `songmh@hit.edu.cn` (M. Song), `xcwang90@gmail.com` (X. Wang)

and Maxwell system. Therefore, much subtle and technical tools have to be used. In particular, the vector-valued elastic source term is decomposed into gradient and curl fields. Subsequently, the far-field pattern is divided into longitudinal and transversal far-field patterns. We establish a one-to-one correspondence between the gradient (respectively curl) and longitudinal (respectively transversal) far-field pattern. Then we derive the stability estimate, which assures that the method is stable and insensitive to the measurement noise. We also note that method is effective and easy to implement.

The rest of the paper is organised as follows. Section 2 describes the modeling of the problem. The uniqueness and stability are studied in Section 3. Section 4 provides numerical examples to illustrate the efficiency of the method.

2. Problem Formulation

We consider the following Navier equation in a homogeneous and isotropic medium in $\mathbb{R}^2$:

$$\mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} + \omega^2\boldsymbol{u} = -\boldsymbol{F},$$

where $\boldsymbol{u}$ is the displacement field, $\boldsymbol{F}$ the body force density, $\omega > 0$ the angular frequency, and λ, μ are the Lamé constants such that $\mu > 0$ and $\lambda + 2\mu > 0$. Throughout this paper we use light and bold fonts for scalar and vector terms, respectively. Let us consider the following scalar and vector curl operators:

$$\mathrm{curl} := (-\partial_2,\ \partial_1), \quad \mathbf{curl} := (\partial_2,\ -\partial_1)^\top.$$

For a bounded domain D in $\mathbb{R}^2$, we consider the spaces $H^1(D)$ and $H(\mathbf{curl};D)$ defined by

$$\begin{aligned} H^1(D) &= \left\{ v \in L^2(D) \mid \nabla v \in \left(L^2(D)\right)^2 \right\}, \\ H(\mathbf{curl};D) &= \left\{ v \in L^2(D) \mid \mathbf{curl} v \in \left(L^2(D)\right)^2 \right\}, \end{aligned}$$

and equip them with the norms

$$\begin{aligned} \|v\|_{H^1(D)} &:= \|v\|_{L^2(D)} + \|\nabla v\|_{L^2(D)}, \\ \|v\|_{H(\mathbf{curl};D)} &:= \|v\|_{L^2(D)} + \|\mathbf{curl} v\|_{L^2(D)}. \end{aligned}$$

Assume that the vector function $\boldsymbol{F}$ is compactly supported in the domain D and has the form

$$\boldsymbol{F} = \nabla f + \mathbf{curl} g \in \left(L^2(D)\right)^2, \tag{2.1}$$

where $f \in H^1(D)$, $g \in H(\mathbf{curl};D)$.

Since

$$\mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} + \omega^2\boldsymbol{u} = \mathbf{0} \quad \text{in} \quad \mathbb{R}^2\backslash\bar{D}, \tag{2.2}$$

any solution $\boldsymbol{u}$ of the Navier equation can be represented in the form

$$\boldsymbol{u} = \nabla\varphi + \mathbf{curl}\psi = \boldsymbol{u}_p + \boldsymbol{u}_s, \tag{2.3}$$