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POSTPROCESSING OF CONTINUOUS GALERKIN SOLUTIONS FOR DELAY DIFFERENTIAL EQUATIONS WITH NONLINEAR VANISHING DELAY

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Abstract. In this paper we propose several postprocessing techniques to accelerate the convergence of the continuous Galerkin solutions for delay differential equations with nonlinear vanishing delay. They are interpolation postprocessings (including integration type, Lagrange type, and polynomial preserving recovery type) and iteration postprocessing. The theoretical expectations are confirmed by numerical experiments.

Key words. Pantograph delay differential equations, quasi-graded mesh, continuous Galerkin methods, postprocessing, global superconvergence.

1. Introduction

Delay differential equations (DDEs) have a wide range of application in science and engineering. The nonlinear vanishing delay equation is an important type of delay differential equation and has received considerable attention in both theoretical analysis and numerical computation since the early 1970s (cf. [14, 15, 10]). Runge-Kutta and collocation methods are two popular numerical methods used to solve this kind functional differential equation, which can be found in the monographs by Bellen and Zennaro [1] and Brunner [3], the survey paper [2], and the recent papers [4, 6, 22, 27], etc..

Finite element methods (FEMs) are efficient numerical methods that extensively used in solving partial differential equations and integral equations. FEMs have also been introduced to solve ordinary differential equations (ODEs) and delay differential equations. See, for example, [8, 9, 20, 21] for ODEs, [7, 16] for DDEs with constant delay, [5, 13] for DDEs with proportional delay, and [26] for Volterra functional integro-differential equations with vanishing delays.

Superconvergence and supercloseness are two hot topics in FEMs. If the errors of numerical solutions U at some points are far less than the global error, we call this phenomenon as superconvergence and the points are called superconvergence points. If the distance between the numerical solution U and some interpolant Πu of the exact solution u is far less than that between the numerical solution U and the exact solution u, that is, $\|U - \Pi u\| \ll \|u - U\|$, we call this phenomenon as supercloseness. Based on the superconvergence and supercloseness, one can put postprocessing techniques onto the numerical solution U and get a new approximation U^* of higher order convergence. There are several popular postprocessing techniques. In the early stages, Sloan iteration was proposed in [23, 24] to improve the convergence of solutions of integral equations. Zienkiewice and Zhu [28, 29] mentioned the postprocessing method of superconvergence patch recovery which leads to global superconvergence of the new approximate solution U^* for partial differential equations (PDEs). The polynomial preserving recovery postprocessing

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method was proposed in [19, 30]. By combining the two adjacent elements and constructing higher order interpolation for one dimensional case (or combining all adjacent elements in high dimensional space), the interpolation postprocessing method ([17, 18]) was proposed to accelerate the numerical solutions.

The superconvergent points of CG and DG solutions are Lobatto [7, 13] and Radau II points [16, 12] respectively for DDEs of constant delay and proportional delay. For DDEs of pantograph type, Huang et al. [12] used two types of postprocessing techniques to improve the global convergence of DG solutions. They [25] obtained all the superconvergent points of CG solutions according to the supercloseness between the CG solution U and the interpolation $\Pi_h u$ of the exact solution under uniform mesh and analyzed the optimal global convergence and local superconvergence of continuous Galerkin solutions for pantograph DDEs under quasi-geometric meshes (more general quasi-graded case).

As a sequel to papers [13, 25], we consider in this paper the delay differential equation with nonlinear vanishing delay,

$$\begin{aligned} &u'(t) = a(t)u(t) + b(t)u(\theta(t)) + f(t), \qquad t \in J = [0, T], \\ &u(0) = u_0. \end{aligned} \tag{1}$$

The delay item θ satisfies the conditions: (i) $\theta(0) = 0$ and $\theta(t) < t$ for $t > 0$, (ii) $\min_{t \in J} \theta^{'}(t) =: q_0 > 0$. We study the superconvergence properties of the "postprocessed" CG solutions obtained by postprocessing for DDE (1). It will be shown that the convergence order of the CG solutions can be improved considerably by several postprocessing methods.

The outline of this paper is as follows. In section 2 we review the CG method for (1) and introduce the convergence results of the CG solutions. In section 3, we illustrate the supercloseness between the CG solution U and a suitable interpolation Πu of the exact solution u and locate all the superconvergent points (subsection 3.1). Then we present two kinds of interpolation postprocessing methods, which respectively based on the supercloseness and the superconvergence points (subsections 3.2 & 3.3). In subsection 3.4, we present a type of postprocessing method using integral iteration to accelerate the convergence order of the CG solutions. In order to obtain higher order of convergence, in section 4, we propose another interpolation postprocessing method based on the superconvergence properties of the nodal points. Finally, we display numerical results to illustrate our theoretical analysis in section 5.

2. The CG method and convergence analysis

In this section, we introduce the CG method for DDE (1) with quasi-graded meshes and the global convergence properties of the CG solution.

2.1. The CG method. We assume that the given functions a, b and f in (1) are continuous on $J = [0, T]$. Suppose that on a small initial subinterval $J_0 = [0, t_0]$ ($t_0 = \theta^k(T), k \in N$), for a suitable value of k, the approximation $\phi(t)$ of the exact solution u is known. $\phi(t)$ can be obtained by the CG method or by the truncation of Taylor expansion of the exact solution $u(t)$. We then solve the following equation

$$\begin{aligned} &u'(t) = a(t)u(t) + b(t)u(\theta(t)) + f(t), \qquad t_0 \le t \le T, \\ &u(t) = \phi(t), \qquad \theta(t_0) \le t \le t_0. \end{aligned} \tag{2}$$