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# The Direct Robin Boundary Value Parabolic System of Time-Resolved Diffuse Optical Tomography with Fluorescence

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**Abstract.** We consider a system of parabolic PDEs with measure data modelling a problem of the time resolved diffuse optical tomography with a fluorescence term and Robin boundary conditions. We focus on the direct problem where the quantity of interest is the density of photons in the diffusion equations and which constitutes a major step to solve the inverse problem of identifiability and reconstruction of diffusion, absorption and concentration of the fluorescent markers. We study the problem under a variational form and its discretization with finite element method and we give some numerical simulation results for verification purpose as well as simulations with real data from a tomograph.

**AMS subject classifications**: 65N21, 65J22, 78A46

**Key words**: Tomography, scattering, fluorescence, contact, non-contact.

## 1 Introduction

We consider the Time-Resolved Diffuse Optical Tomography (TR-DOT) problem with fluorescence arising in the medical and biological imaging for soft tissues, small animal studies and noninvasive methods of tumor detection [1, 9, 13, 14]. The final goal in this imaging modalities is the reconstruction of physical parameters such as diffusion and absorption coefficients. While IRM and X-Ray tomography are usually of high resolution and deep penetration properties allowing exploring the object in-depth but with poor

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contrast, the optical tomography is very sensitive to the contrast but with poor resolution. In addition, the optical tomography is only efficient in thin regions close to the detectors. However, due to its cheap cost and noninvasive features, its recommended for some type of tumor (skin, breasts, neonatal imaging,...). In order to increase its efficiency, it is usually coupled with the use of fluorescent molecules which enhance the contrast in the tissues. The mathematical model of (TR-DOT) is a parabolic system with a memory term and measure data. In simple geometric setting (disk, cylinder) and assuming some regularity of the solutions, the mathematical study of the resulting problem follows from standard methods of the calculus of variations but in the general setting one has to use more technical and involved methods to solve the parabolic systems (e.g. approximation approach [2, 3, 12]). Moreover, there are two different experimental setting leading to two kind of data and measurements: the contact case where the detection and excitation fibers are set directly on the object under consideration and the non-contact case where they are not. This yields two mathematical models for which the governing equations are unchanged while the data and measurement terms are either Dirac masses on isolated points in the first case or Dirac masses on lines in 2D and surfaces in 3D configurations (any all cases sets of small dimension). The main goal in this article is to study such problems and to perform the numerical analysis as well as preliminary numerical simulations on both academic data for verification purpose in different implementation settings (Matlab and Feel++ softwares) as well as real data from an experimental tomograph of Icube Laboratory (university of Strasbourg). The diffusion approximation for the optical tomography called time resolved diffuse optical tomography, DOT-TR, is widely used and documented in medical and biological imaging, we refer the reader to the review of S. Arridge and the references therein [1]. In contrast, only a few references are available for the fluorescent case and to our knowledge almost no mathematical and numerical study have been done yet. We refer to [4] for the physical model and related references on the subject. The paper is organized as follows. The Section 2 proposes a study for the well-posedness of the coupled system of equations for the contact case, Section 2.1 and the non-contact case, Section 2.2. Some mathematical preliminaries are introduced, the weak formulations and some theoretical results are stated. In Section 3, we set the discrete framework with emphasis on the methods to handle the fluorescence source term which can be seen as a memory source term, but can also be treated without explicit integration (Duhamel formulae) by introducing an Ordinary Differential Equations (ODE) to the initial system. Finally, two convergence results are given depending on the chosen source term for the contact and the non-contact mode.

## 2 Mathematical model-Weak formulations

In this section, we briefly give some mathematical tools and results on the models under consideration. We derive also the associated weak formulations on which our discretization is based. We refer for details to [4]. We call tomograph an experimental device which