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Point Integral Method for Solving Poisson-Type Equations on Manifolds from Point Clouds with Convergence Guarantees

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Abstract. Partial differential equations (PDE) on manifolds arise in many areas, including mathematics and many applied fields. Due to the complicated geometrical structure of the manifold, it is difficult to get efficient numerical method to solve PDE on manifold. In the paper, we propose a method called point integral method (PIM) to solve the Poisson-type equations from point clouds. Among different kinds of PDEs, the Poisson-type equations including the standard Poisson equation and the related eigenproblem of the Laplace-Beltrami operator are one of the most important. In PIM, the key idea is to derive the integral equations which approximates the Poisson-type equations and contains no derivatives but only the values of the unknown function. This feature makes the integral equation easy to be discretized from point cloud. In the paper, we explain the derivation of the integral equations, describe the point integral method and its implementation, and present the numerical experiments to demonstrate the convergence of PIM.

AMS subject classifications: 65N12, 65N25, 65N75

Key words: Point integral method, point cloud, Laplace-Beltrami operator, convergence.

1 Introduction

Partial differential equations (PDE) on manifolds arise in many areas, including geometric flows along manifolds in geometric analysis [8], movements of particles confined to surfaces in quantum mechanics [9, 29], and distributions of physical or chemical quantities along interfaces in fluid mechanics [10], among others. It is well-known that one can extract the geometric information of the manifolds by studying the behavior of partial differential equations or differential operators on the manifolds. This observation has been exploited both in mathematics, especially geometric analysis [39], and in applied

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fields, including machine learning [3,17], data analysis [28], computer vision and image processing [21], geometric processing of 3D shapes [19,25,26]. Poisson equation on manifolds and the related eigenproblem of the Laplace-Beltrami operator are one of the most important, and have found applications in many fields. For instance, the eigensystem of the Laplace-Beltrami operator has been used for representing data in machine learning for dimensionality reduction [2], and for representing shapes in computer vision and computer graphics for the analysis of images and 3D models [25,26].

In this paper, we propose a method to solve the Poisson equations on manifolds from point clouds with convergence guarantees. Unlike a mesh or a Euclidean grid, which may be difficult to generate or may introduce extra complexity, point cloud is the simplest way of representing a manifold, which is often made ready for use in practice and whose complexity depends only on the manifold itself. The main observation is that the Poisson equations can be approximated by certain integral equations which can be easily discretized and has a faithful approximation from point clouds. More precisely, we consider the Poisson equation with Neumann boundary condition:

$$\begin{cases} -\Delta u(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x}\in\mathcal{M}, \\ \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x}\in\partial\mathcal{M}, \end{cases} \tag{1.1}$$

where $\mathcal{M}$ is a k dimensional submanifold isometrically embedded in $\mathbb{R}^d$. We show that its solution is well approximated by the solution of the following integral equation:

$$\begin{aligned} &-\frac{1}{t}\int_{\mathcal{M}}(u(\mathbf{x})-u(\mathbf{y}))R\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right)\mathrm{d}\mathbf{x} \\ &=\int_{\mathcal{M}} f(\mathbf{x})\bar{R}\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right)\mathrm{d}\mathbf{x}+2\int_{\partial\mathcal{M}} g(\mathbf{x})\bar{R}\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right)\mathrm{d}\mathbf{x}, \end{aligned} \tag{1.2}$$

where the function $R(r):\mathbb{R}^+\to\mathbb{R}^+$ is either compactly supported or decays exponentially and

$$\bar{R}(r) = \int_r^{+\infty} R(s)\mathrm{d}s. \tag{1.3}$$

One choice of the function R is the well-known Gaussian. As the integral equation involves no derivatives of the unknown function u but only the function values, it can be easily discretized from a point cloud which samples the underlying manifold. We call this method point integral method (PIM) as it only requires the approximation of integrals from the discrete point clouds. It has been shown that PIM has convergence guarantees for solving the Poisson-type equations on manifolds. The readers who are interested in the convergence analysis are referred to our companion papers [32–34]. In this paper, we focus on describing the point integral method and its implementation, and presenting the numerical experiments to demonstrate the convergence of PIM.