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Lyapunov-type Inequalities for a System of Nonlinear Differential Equations

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Abstract: This paper presents several new Lyapunov-type inequalities for a system of first-order nonlinear differential equations. Our results generalize and improve some existing ones.

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1 Introduction

In this paper we are concerned the following system of nonlinear differential equations:

$$\begin{cases} x'(t) = \alpha(t)x(t) + \beta(t)f(y(t)), \\ y'(t) = -\gamma(t)g(x(t)) - \alpha(t)y(t), \end{cases} \tag{1.1}$$

where $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are real-valued piece-wise continuous functions defined on $\mathbf{R}$, f and g are real-valued continuous functions defined on $\mathbf{R}$.

If $f(y) \equiv y$ and $g(x) \equiv x$, then the system (1.1) reduces to the first-order linear Hamiltonian system

$$\begin{cases} x'(t) = \alpha(t)x(t) + \beta(t)y(t), \\ y'(t) = -\gamma(t)x(t) - \alpha(t)y(t). \end{cases} \tag{1.2}$$

Note that (1.1) contains many well-known and well studied differential equations as special cases. For example, the following second-order linear differential equations can be written in form of (1.1)

$$x''(t) + q(t)x(t) = 0, \tag{1.3}$$

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$$x''(t) + p(t)x'(t) + q(t)x(t) = 0 \tag{1.4}$$

and

$$[\rho(t)x'(t)]' + q(t)x(t) = 0, \tag{1.5}$$

where $p(t)$ and $q(t)$ are real-valued piece-wise continuous functions defined on $\mathbf{R}$ and $\rho(t)$ is a real-valued continuous function defined on $\mathbf{R}$ with $\rho(t) > 0$. If we let

$$y(t) = \exp\left\{\int_0^t p(s)\mathrm{d}s\right\} x'(t),$$

then (1.4) can be written in the form of (1.1) with

$$f(y) \equiv y, \qquad g(x) \equiv x,$$

and

$$\alpha(t) = 0, \quad \beta(t) = \exp\left\{-\int_0^t p(s)\mathrm{d}s\right\}, \quad \gamma(t) = -q(t)\exp\left\{\int_0^t p(s)\mathrm{d}s\right\}.$$

In 1907, Lyapunov[1] established the well known inequality which provides a lower bound for the distance between two consecutive zeroes a, b of the solution of the second-order linear differential equation (1.3), namely

$$(b-a)\int_a^b |q(t)|\mathrm{d}t > 4. \tag{1.6}$$

Since then many improvements on (1.6) have been developed and similar inequalities have been obtained for other types of differential equations (see [2] and [3]). For instance, Wintner[4] showed that if (1.3) has a real solution $x(t)$ such that

$$x(a) = x(b) = 0, \quad x(t) \neq 0, \qquad t \in (a, b), \tag{1.7}$$

then

$$(b-a)\int_a^b q^+(t)\mathrm{d}t > 4, \tag{1.8}$$

where and in the sequel $q^+(t) = \max\{q(t), 0\}$, $a, b \in \mathbf{R}$ with $a < b$. Moreover, Wintner proved that the constant 4 cannot be replaced by a larger number. Later Hartman and Wintner[5] established

$$\int_a^b q^+(t)(t-a)(b-t)\mathrm{d}t > b-a. \tag{1.9}$$

In 1969, Fink and Mary[6] extended Lyapunov inequality (1.8) to (1.4) and obtained the following Lyapunov-type inequality

$$(b-a)\int_a^b q^+(t)\mathrm{d}t > 4\exp\left\{-\frac{1}{2}\int_a^b |p(t)|\mathrm{d}t\right\}. \tag{1.10}$$

In 2003, Yang[7] extended Lyapunov inequality (1.8) to the second-order differential equation (1.5) and obtained the following inequality

$$\int_a^b q^+(t)\mathrm{d}t \int_a^b [\rho(t)]^{-1}\mathrm{d}t > 4,$$

if (1.5) has a solution $x(t)$ satisfying (1.7). In 2003, Guseinov and Kaymakcalan[8] further generalized (1.8) to the planar linear Hamiltonian system (1.2) and derived the following Lyapunov-type inequality

$$\int_a^b |\alpha(t)|\mathrm{d}t + \left[\int_a^b \beta(t)\mathrm{d}t \int_a^b \gamma^+(t)\mathrm{d}t\right]^{\frac{1}{2}} \geq 2, \tag{1.11}$$