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An Equivalent Characterization of $CMO(\mathbb{R}^n)$ with A_p Weights

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Abstract. Let $1<p<\infty$ and $\omega\in A_p$. The space $CMO(\mathbb{R}^n)$ is the closure in $BMO(\mathbb{R}^n)$ of the set of $C_c^\infty(\mathbb{R}^n)$. In this paper, an equivalent characterization of $CMO(\mathbb{R}^n)$ with A_p weights is established.

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1 Introduction

The goal of this paper is to provide an equivalent characterization of $CMO(\mathbb{R}^n)$, which is useful in the study of compactness of commutators of singular integral operator and fractional integral operator.

The space $BMO(\mathbb{R}^n)$ is defined by the set of functions $f\in L^1_{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{BMO(\mathbb{R}^n)}:=\sup_{Q\subset\mathbb{R}^n} M(f,Q)<\infty,$$

where

$$M(f,Q):=\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx,\quad f_Q:=\frac{1}{|Q|}\int_Q f(x)dx.$$

The space $CMO(\mathbb{R}^n)$ is the closure in $BMO(\mathbb{R}^n)$ of the set of $C_c^\infty(\mathbb{R}^n)$, which is a proper subspace of $BMO(\mathbb{R}^n)$.

In fact, it is known that $CMO(\mathbb{R}^n)=VMO_0(\mathbb{R}^n)$, where $VMO_0(\mathbb{R}^n)$ is the closure of $C_0(\mathbb{R}^n)$ in $BMO(\mathbb{R}^n)$, see [2,3,9]. Here $C_0(\mathbb{R}^n)$ is the set of continuous functions on $\mathbb{R}^n$ which vanish at infinity. Neri [8] gave a characterization of $CMO(\mathbb{R}^n)$ by Riesz transforms. Meanwhile, Neri proposed the following characterization of $CMO(\mathbb{R}^n)$ and its proof was established by Uchiyama in his remarkable work [11].

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Theorem 1.1. *Let $f\in BMO(\mathbb{R}^n)$. Then $f\in CMO(\mathbb{R}^n)$ if and only if f satisfies the following three conditions*

(a) $\lim_{a\to 0}\sup_{|Q|=a} M(f,Q)=0$;

(b) $\lim_{a\to \infty}\sup_{|Q|=a} M(f,Q)=0$;

(c) $\lim_{|x|\to\infty} M(f,Q+x)=0$ *for each cube* $Q\subset\mathbb{R}^n$, *where* $Q+x:=\{y+x:y\in Q\}$.

Recently, Guo, Wu and Yang [6] established an equivalent characterization of space $CMO(\mathbb{R}^n)$ by local mean oscillations. Lots of works about space $CMO(\mathbb{R}^n)$ have been studied, see [4] for example. Muckenhoupt and Wheeden [7, Theorem 5] showed the norm of $BMO_\omega(\mathbb{R}^n)$(see Definition 1.2) is equivalent to the norm of $BMO(\mathbb{R}^n)$, where the weight function ω is Muckenhoupt A_p weight. So it is natural to consider equivalent characterizations of $CMO(\mathbb{R}^n)$ associated to A_p weights.

To state our main results, we first recall some relevant notions and notations.

The following class of A_p was introduced in [1,5].

Definition 1.1. *Let $\omega(x)\geq 0$ and $\omega(x)\in L^1_{loc}(\mathbb{R}^n)$. For $1<p<\infty$, we say that $\omega(x)\in A_p$ if there exists a constant $C>0$ such that for any cube Q,*

$$\Big(\frac{1}{|Q|}\int_Q \omega(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q \omega(x)^{-\frac{1}{p-1}}dx\Big)^{p-1}\leq C. \tag{1.1}$$

Also, for $p=1$, we say that $\omega(x)\in A_1$ if there is a constant $C>0$ such that

$$M\omega(x)\leq C\omega(x), \tag{1.2}$$

where M is the Hardy-Littlewood maximal operator. For $p\geq 1$, the smallest constant appearing in (1.1) *and* (1.2) *is called the A_p characteristic constant of ω and is denoted by $[\omega]_{A_p}$.*

Definition 1.2. *Let $\omega\in A_p$. For a cube Q in $\mathbb{R}^n$, we say a function $f\in L^1_{loc}(\mathbb{R}^n)$ is in $BMO_\omega(\mathbb{R}^n)$ if f satisfies*

$$\|f\|_{BMO_\omega(\mathbb{R}^n)}:=\sup_{Q\subset\mathbb{R}^n} M(f,Q)_\omega<\infty,$$

where

$$m(f,Q)_\omega:=\frac{1}{\omega(Q)}\int_Q f(x)\omega(x)dx,$$
$$M(f,Q)_\omega:=\frac{1}{\omega(Q)}\int_Q |f(x)-m(f,Q)_\omega|\omega(x)dx.$$