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A Simple Semi-Implicit Scheme for Partial Differential Equations with Obstacle Constraints

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Abstract. We develop a simple and efficient numerical scheme to solve a class of obstacle problems encountered in various applications. Mathematically, obstacle problems are usually formulated using nonlinear partial differential equations (PDE). To construct a computationally efficient scheme, we introduce a time derivative term and convert the PDE into a time-dependent problem. But due to its nonlinearity, the time step is in general chosen to satisfy a very restrictive stability condition. To relax such a time step constraint when solving a time dependent evolution equation, we decompose the nonlinear obstacle constraint in the PDE into a linear part and a nonlinear part and apply the semi-implicit technique. We take the linear part implicitly while treating the nonlinear part explicitly. Our method can be easily applied to solve the fractional obstacle problem and min curvature flow problem. The article will analyze the convergence of our proposed algorithm. Numerical experiments are given to demonstrate the efficiency of our algorithm.

AMS subject classifications: 65N06, 65N12, 35J60

Key words: Numerical methods, nonlinear elliptic equations, obstacle problem, semi-implicit scheme.

1. Introduction

In this paper, we develop efficient semi-implicit schemes to a class of obstacle problems as stated as follow [2, 5–7, 9, 25]. For a given energy functional $E(u)$, we determine $u \in K$ such that $E(u) = \inf_{v\in K} E(v)$ for some $K = \{v \in H^1 | v \geq \psi \text{ in } \Omega, \ v = g \text{ on } \partial\Omega\}$. The function ψ is a given obstacle function, Ω is the computational domain and g is the boundary condition. This class of problems can be found in various fields including the classical problem of elastic membrane modeling, pricing model in

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financial mathematics, porus media computations, computing the torsion of an elastic-plastic cylinder, Stefan problems for crystal growth simulation, min curvature flow in image processing [20], and etc. For example, in the problem of elastic membrane constrained on obstacle, the potential energy $E(u)$ is given by

$$E(u)=\int_\Omega \frac{1}{2}|\nabla u|^2 - fud\mathbf{x}, \tag{1.1}$$

where f is external force on u. For the minimal surface obstacle problem, the potential energy is proportional to its surface area and it leads to the energy functional

$$E(u)=\int_\Omega \sqrt{1+|\nabla u|^2} - fud\mathbf{x}.$$

One possible numerical approach to this class of problems is the projected relaxation method [10, 28] which first reformulates the problem using elliptic variational inequalities [13]. This class of methods is easy to implement and is proven to be convergent. However, its convergence speed depends on the relaxation parameter and the convergent might be slow in practice. To accelerate the algorithms, the multigrid method has been adopted as discussed in [1, 13, 17, 30].

Another way to solve the obstacle problem is via the optimization formulation. In [16], a Langrange multiplier is used to incorporate the constraint in the functional. In [24], a penalty term is introduced in the functional to encourage the solution to satisfy the constraint. The solution obtained by this method is not exact and the penalty parameter needs to be very small, of $\mathcal{O}(h^{-2})$. In [27], an L^1 penalty is added to the functional to relax the constraint of the obstacle. The equivalence of their formulation to (1.1) is proven [8, 21]. A related splitting Bregman algorithm has recently been implemented in [12, 27]. Note that the efficiency of this method depends on the application and also on the choice of the parameters. For linear problems, i.e., when the operator A is linear, this method converges very fast. But for nonlinear problems, i.e., in cases when we do not have any fast algorithm to invert A, the overall algorithm can be less efficient. Anther constraint approach has been developed in [29,31] which iteratively identifies the subdomain where the constraint is active. For the region where the constraint is inactive, the method recomputes the solution to the corresponding Euler-Lagrange equation of the functional. An augmented Lagrangian active set method has been proposed in [19] which takes advantage of the primal-dual formulation of the discretized obstacle problem. In [33], a primal-dual hybrid gradient method is also developed to solve the obstacle problem. Since the starting point of most of these optimization methods is an energy form, they might not be able to easily extend to fractional obstacle problems. Moreover, those methods are designed to solve an optimization problem, they cannot be applied to flow problems where intermediate steps contain the time evolution of the solution such as the min curvature flow problem.

In this work, we determine the minimizer of the variational problem by solving the corresponding Euler-Lagrange equation. In particular, we solve

$$\min(Au-f,u-\psi)=0 \quad \text{on} \quad \Omega \tag{1.2}$$