J. Math. Study
doi: 10.4208/jms.v53n2.20.04

Vol. **53**, No. 2, pp. 192-211
June 2020

Efficient Laguerre and Hermite Spectral Methods for Odd-order Differential Equations in Unbounded Domains

Cheng Xu, Xuhong Yu and Zhongqing Wang*

School of Science, University of Shanghai for Science and Technology, Shanghai 200093, China.

Received March 26, 2019; Accepted July 7, 2019;
published online April 20, 2020.

Dedicated to Professor Jie Shen on the Occasion of his 60th Birthday

Abstract. Laguerre dual-Petrov-Galerkin spectral methods and Hermite Galerkin spectral methods for solving odd-order differential equations in unbounded domains are proposed. Some Sobolev bi-orthogonal basis functions are constructed which lead to the diagonalization of discrete systems. Numerical results demonstrate the effectiveness of the suggested approaches.

AMS subject classifications: 65N35, 33C45, 35J58

Key words: Dual-Petrov-Galerkin spectral methods, Laguerre functions, Hermite functions, Sobolev bi-orthogonal functions, odd-order differential equations.

1 Introduction

Spectral methods possess high-order accuracy and play an important role in scientific and engineering computations, see [2–4, 6, 20, 21] and the references therein. For problems set in unbounded domains, such as fluid flows in an infinite strip, nonlinear wave equations in quantum mechanics and so on, the direct and commonly used spectral approaches are based on orthogonal systems on infinite intervals, i.e., the Hermite and Laguerre orthogonal polynomials/functions. There exist a number of investigations on Laguerre and Hermite spectral methods for second and higher even-order equations, see [1, 7, 8, 11, 12, 18, 25, 26, 28, 29]. However, some physically interesting equations, e.g., the Korteweg-de Vries equation, are odd-order equations. It is noteworthy that relatively few studies have focused on odd-order equations. This is partly due to the facts that the

*Corresponding author. *Email addresses:* zqwang@usst.edu.cn (Z. Wang), xhyu@usst.edu.cn (X. Yu), qfkxu@foxmail.com (C. Xu)

usual spectral Galerkin or collocation methods for odd-order problems lead to the condition number of algebraic systems growing too fast, and often exhibit unstable modes, cf. [10, 17].

Since the main differential operators in odd-order differential equations are not symmetric, it is reasonable to use the Petrov-Galerkin spectral method to solve this kind of problems. Recently, Ma and Sun [14, 15] developed a stable Legendre-Petrov-Galerkin and Chebyshev collocation method for the third-order differential equations in bounded domains. Shen [19] proposed an efficient Legendre dual-Petrov-Galerkin spectral method for the third and higher odd-order equations in bounded domains. Shen and Wang [23] also presented Legendre and Chebyshev dual-Petrov-Galerkin spectral methods for the first-order hyperbolic equations in bounded domains. For semi-infinite interval, Shen and Wang [22] considered a Laguerre dual-Petrov-Galerkin spectral method for the Korteweg-de Vries equation, which led to a strongly coercive and easily invertible linear system. It is pointed out that, the resulting linear systems mentioned above is sparse or compactly sparse. However, in most cases, people always hope to get a completely diagonalized algebraic system.

As it's known that, the Fourier system $\{\exp(ik\cdot)\}_{k\in\mathbb{Z}}$ is the most desirable basis owing to the facts: (i) the availability of Fast Fourier Transform; and (ii) the Sobolev orthogonality, which makes the corresponding algebraic system completely diagonal. However, the Fourier spectral method is only available for periodic problems. For non-periodic problems, the usual spectral methods merely get a sparse rather than diagonal system. Recently, Liu *et al.* [11–13] considered the diagonalized Laguerre and Hermite spectral methods for even-order problems in unbounded domains. Motivated by the works [11–13, 19, 24] and by those on Sobolev orthogonal basis functions [5,16], the main purpose of this paper is to construct Sobolev bi-orthogonal Laguerre and Hermite basis functions, and propose efficient Laguerre dual-Petrov-Galerkin and Hermite Galerkin spectral methods for odd-order problems.

The main advantages of the suggested algorithms include:

- the approximate solutions can be represented as truncated Fourier-like series;
- the resulting linear systems are diagonal and the condition numbers are equal to one;
- the computational cost is much lower than that of the classical Laguerre/Hermite spectral methods.

The remainder of this paper is organized as follows. In Section 2, we introduce the generalized Laguerre and Hermite functions. In Section 3, we construct two kinds of Sobolev bi-orthogonal generalized Laguerre functions corresponding to the third and fifth order differential equations on the half line, and propose the diagonalized Laguerre dual-Petrov-Galerkin spectral methods. In Section 4, we construct two kinds of Sobolev bi-orthogonal generalized Hermite functions corresponding to the third and fifth order differential equations on the whole line, and propose the diagonalized Hermite Galerkin