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Long Time Well-Posedness of the MHD Boundary Layer Equation in Sobolev Space

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Abstract. In this paper, we study the long time well-posedness of 2-D MHD boundary layer equation. It was proved that if the initial data satisfies

$$\|(u_0, h_0-1)\|_{H^{3,0}_\mu \cap H^{1,2}_\mu} \le \varepsilon,$$

then the life span of the solution is at least of order $\varepsilon^{2-\eta}$ for $\eta > 0$.

Key Words: MHD boundary layer equation, Sobolev space, well-posedness.

AMS Subject Classifications: 35Q30, 76D03

1 Introduction

In this paper, we study the well-posedness of the MHD boundary layer equation in $\mathbf{R}^2_+$:

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u - h\partial_x h - g\partial_y h = \kappa\partial_y^2 u - \partial_x p, \\ \partial_t h + \partial_y(vh - ug) = \nu\partial_y^2 h, \\ \partial_x u + \partial_y v = 0, \quad \partial_x h + \partial_y g = 0, \\ (u, v, \partial_y h, g)|_{y=0} = 0 \quad \text{and} \quad \lim\limits_{y\to+\infty}(u,h) = (U(t,x), H(t,x)), \\ (u,h)|_{t=0} = (u_0, h_0), \end{cases} \tag{1.1}$$

where (u,v) denotes the velocity field of the boundary layer flow, (h,g) denotes the magnetic field, and $(U(t,x), H(t,x), p(t,x))$ denotes the outflow of velocity, magnetic and

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pressure, which satisfies the Bernoulli's law:

$$\partial_t U + U\partial_x U - H\partial_x H + \partial_x p = 0, \quad \partial_t H + U\partial_x H - H\partial_x U = 0.$$

This system is a boundary layer model, which describes the behaviour of the solution of the viscous MHD equations when the viscosity and the resistivity tend to zero [6,11].

When $h = 0$, the system (1.1) is reduced to the classical Prandtl equation:

$$\partial_t u + u\partial_x u + v\partial_y u = \kappa\partial_y^2 u - \partial_x p, \quad \partial_x u + \partial_y v = 0.$$

The well-posedness theory of the 2-D Prandtl equation was well understood. For the monotonic data, Oleinik [14] proved the local existence and uniqueness of classical solutions. With the additional favorable pressure, Xin and Zhang [16] proved the global existence of weak solutions of the Prandtl equation. Sammartino and Calflisch [15] established the local well-posedness of the Prandtl equation for the analytic data. Recently, Alexandre et al. [1] and Masmoudi-Wong [13] independently developed direct energy method to prove the well-posedness of the Prandtl equation for monotonic data in Sobolev spaces. Without monotonicity, Gérard-Varet and Dormy [7] proved the ill-posedness of the Prandtl equation in Sobolev space. However, the Prandtl equation is well-posed in Gevrey class 2 for a class of non-monotone data with non-degenerate critical points [4,8,12]. On the other hand, E and Engquist [5] proved that the analytic solution can blow up in a finite time [5]. See [9] for the extension to van Dommelen-Shen type singularity. For small analytic initial data, Zhang and the fourth author [18] proved the long time well-posedness of the Prandtl equation: if the initial data satisfies

$$\|e^{\frac{1+y^2}{8}} e^{|D_x|} u_0\|_{B^{\frac{1}{2},0}} \leq \varepsilon,$$

then the lifespan of the solution is greater than $\varepsilon^{-\frac{4}{3}}$. In [10], Ignatova and Vicol obtained a larger lifespan $\exp \frac{\varepsilon^{-1}}{\ln \varepsilon^{-1}}$ with small analytical data of size $\mathcal{O}(\varepsilon)$, whose analytical width $\tau_\varepsilon \to \infty$ as $\varepsilon \to 0$.

In two recent interesting works [6, 11], the authors showed that the tangential magnetic field has stabilization effect on the boundary layer of the fluid. In particular, they proved the well-posedness of the system (1.1) for the data without monotonicity under an uniform tangential magnetic field.

The goal of this paper is two folds: (1) present a simple proof of well-posedness based on the paralinearization method developed in [3]; (2) study the long time well-posedness of the system (1.1) for small data in Sobolev space. In [17], Xu and Zhang proved a long time existence of the Prandtl equation for the data close a monotonic shear flow. However, it is unclear how the lifespan of the solution depends on the data. Here we would like to give the explicit lifespan of the solution of the system (1.1).

For simplicity, we consider a uniform outflow $(U, H) = (0, 1)$ and take $\kappa = \nu = 1$. Let