J. Math. Study
doi: 10.4208/jms.v53n3.20.01

Vol. **53**, No. 3, pp. 237-246
September 2020

Elliptic Problems in Curved Domains Using Cubature Points Based Triangular Spectral Elements and Isoparametric Mappings

Richard Pasquetti*

Lab. J. A. Dieudonné, Université Côte d'Azur, CNRS, Inria, F-06108 Nice, France.

Received March 25, 2019; Accepted May 31, 2019;
Published online May 21, 2020.

Abstract. Using the cubature points based triangular spectral element method and isoparametric mappings, we provide accuracy results for elliptic problems in non polygonal domains. Two regimes of convergence, associated to the bulk and to the boundary of the computational domain are clearly discerned and an efficient way to define the isoparametric mapping is proposed.

AMS subject classifications: 65N30, 65N35

Key words: Spectral element method, triangular elements, curved domains, isoparametric mappings, cubature points, diagonal mass matrix.

---

1 Introduction

The present paper extends the study presented in [14], concerned with high order Finite Element Methods (FEMs) in non polygonal domains, in order (i) to provide a deeper analysis of the accuracy results and also (ii) to propose a way to improve them. More precisely, in [14] it was pointed out that when isoparametric mappings are used to approximate a curved boundary, then the convergence of the approximate solution to the exact one, with respect to the polynomial degree, shows two regimes, the first one being associated to the bulk of the domain and the other being controlled by the boundary part. Here we want to clarify this point. Moreover, even if focusing on a particular type of isoparametric mapping, a curved boundary can be approximated in many different ways. This paper provides a way to define the mapping that yields better accuracy results than the way used in [14].

*Corresponding author. *Email address:* `richard.pasquetti@unice.fr` (R. Pasquetti)

As in [14], the study is restricted to isoparametric mappings, i.e., we do not consider different treatments of curved domains like those involved with transfinite mappings, with the Isogeometric FEM or with the Nurbs Enhanced FEM, see [15] and references herein. Despite the fact that isoparametric mappings show some failures, especially in general an approximate description of the boundary, they indeed remain simple to implement and moreover preserve some important properties of Spectral Element Methods (SEMs), like the diagonal feature of the mass matrix. Moreover, from results first presented in [12], where different isoparametric mappings are compared, we restrict ourselves to isoparametric mappings based on the resolution of Partial Differential Equations (PDEs), namely the Laplace equation (harmonic extension) or the equation of linear elasticity.

With respect to some other high order FEMs developed for triangular elements, e.g., the Fekete-Gauss triangular spectral element method (*T*SEM) [11], the Fekete (resp. Gauss) points of the triangle being used as interpolation (resp. quadrature) points, as in [14] here we use the "Cubature *T*SEM", i.e., only based on the so-called cubature points, see [2,4,7,10]. Because with the Cubature *T*SEM the interpolation and quadrature points coincide, the mass matrix is diagonal, so that this interesting property of the usual SEM is recovered. Indeed, in quadrangles the usual SEM makes use of the tensorial product of Gauss-Lobatto-Legendre (GLL) points both for interpolations and quadratures. This may be useful, e.g., to address evolution problems with explicit time schemes or to easily define high order differentiation operators [9]. Indeed, in both cases the required inversion of the mass matrix can be trivially achieved.

High order FEMs are generally based on a polynomial approximation of total degree N in each triangular element, the number of degrees of freedom being then equal to $(N+1)(N+2)/2$. However, it is known that in the frame of Lagrangian methods one cannot find a single set of $n=(N+1)(N+2)/2$ points, with $3N$ points on the triangle boundary and $(N-1)(N-2)/2$ inside, such that an enough powerful quadrature rule exists [5,16]. To overcome this difficulty, as first suggested in [2] one should enrich the space $\mathbb{P}_N(\hat{T})$, of the polynomials of total degree N defined on the reference triangle (master element) $\hat{T}$, with bubble functions of degree $N'>N$, i.e., to make use of the space $\mathbb{P}_N U b \mathbb{P}_{N'-3}$ where b is the unique bubble function of $\mathbb{P}_3(\hat{T})$. This allows to increase the number of inner points up to $(N'-1)(N'-2)/2$ while keeping equal to $3N$ the number of boundary points. Up to now, satisfactory results have been obtained with $N'=N+1$ for $1<N<5$, $N'=N+2$ for $N=5$ and $N'=N+3$ for $5<N<9$ [7]. Such values of N' are those obtained when requiring that the integrals of polynomials of degree $N+N'-2$ are exactly computed, from a theoretical result that traces back to the 70's [1], but they could be decreased from the less demanding criterion recently proposed in [3].

Note that when using the so-called "condensation technique", one first computes the solution at the nodes located at the sides of the triangular elements and then locally its inner point values. As a result, the size of the algebraic system increases linearly with N, rather than in N^2. Moreover, the system only shows a $O(N)$ condition number and using $N'>N$ yields a negligible increase of the computational cost. Although using the Cubature *T*SEM, with implementation of the condensation technique, the results of the