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Asymptotic Decomposition for Nonlinear Damped Klein-Gordon Equations

Ze Li[1,*] and Lifeng Zhao[2]

[1] *School of Mathematics and Statistics, Ningbo University, Ningbo 315211, China;*
[2] *School Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China.*

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Abstract. In this paper, we prove that if the solution to the damped focusing Klein-Gordon equations is global forward in time with bounded trajectory, then it will decouple into the superposition of divergent equilibriums. The core ingredient of our proof is the existence of the "concentration-compact attractor" introduced by Tao which yields a finite number of asymptotic profiles. Using the damping effect, we can prove all the profiles are equilibrium points.

AMS subject classifications: 35Q60, 37N20, 35L10.

Key words: Nonlinear Klein-Gordon equations, damping, soliton resolution, global attractor.

1 Introduction

In this paper, we consider the following damped focusing Klein-Gordon equation:

$$\begin{cases} u_{tt}-\Delta u+u+2\alpha u_t-|u|^{p-1}u=0, \\ (u(0,x),\partial_t u(0,x))=(u_0(x),u_1(x))\in\mathcal{H}, \end{cases} \tag{1.1}$$

where $\mathcal{H}=H^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$, $\alpha\geq 0$. The energy is given by

$$E(f,g)=\int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla f|^2+\frac{1}{2}|f|^2+\frac{1}{2}|g|^2-\frac{1}{p+1}|f|^{p+1}\right)dx.$$

Dynamical systems of the type (1.1) appear in a number of physical settings, for example it describes the behavior of waves propagating in a nonlinear medium with damping effect, see [25,26].

*Corresponding author. *Email addresses:* `rikudosennin@163.com` (Z. Li), `zhaolf@ustc.edu.cn` (L. Zhao)

We focus on the dynamic behavior of solutions to (1.1) in this paper. When $\alpha=0$, (1.1) is called nonlinear Klein-Gordon equation (NLKG). When $\alpha>0$, we call (1.1) the nonlinear damped Klein-Gordon equation (dNLKG). For NLKG, Cazenave [3] obtained the following dichotomy: solutions either blow up at finite time or are global forward in time and bounded in $\mathcal{H}$, provided $1<p<\infty$, when $d=1$, $1<p\leq 5$, when $d=2$ and $1<p\leq \frac{d}{d-2}$ when $d\geq 3$. For dNLKG, Feireisl [17] gave an independent proof of the boundedness of the trajectory to global solutions, for $1<p<1+\frac{d}{d-2}$ when $d\geq 3$. For dNLKG, $1<p<1+\frac{2}{d-2}$, $1\leq d\leq 6$, Burq, Raugel, Schlag [2] proved that radial global solutions will converge to equilibrium points as time goes to infinity. A natural problem is what happens for non-radial solutions? It is widely conjectured that the solutions will decouple into the superposition of equilibrium points. In the positive direction, Feireisl [17] constructed a global solution to dNLKG which decouples into the superposition of finite number of divergent shifted ground states. Indeed, this problem is closely related to the soliton resolution conjecture in dispersive equations. The (imprecise sense) soliton resolution conjecture states that for "generic" large global solutions, the evolution asymptotically decouples into the superposition of divergent solitons, a free radiation term, and an error term tending to zero as $t\to\infty$. For more expression and history on the soliton resolution, see Soffer [32].

There are a lot of works devoted to the verification of the soliton resolution conjecture. Duyckaerts, Kenig, and Merle [10] first made a breakthrough on this topic. For radial data to three dimensional focusing energy-critical wave equations, they proved the solution with bounded trajectory asymptotically decouples into the superposition of a finite number of rescaled ground states plus a radiation term. One of the key ingredient of their arguments is the novel tool, called "channels of energy" introduced by [10, 11]. The method developed by them has been applied to many other situations, such as [7,8,22–24] for wave maps, [6, 13, 20, 30] for semilinear wave equations. By a weak version of channel energy, the soliton resolution along a sequence of times for radial even dimensional critical wave equations and energy critical equivariant geometric wave equations such as Yang-Mills, wave maps was proved by [5, 6, 21]. Recently Duyckaerts, Jia, Kenig, Merle [14, 15] solved the soliton resolution along a sequence of times for the focusing energy critical wave equation.

It is known that (1.1) admits a ground state which is the radial positive stationary solution with the minimized energy among all the non-zero stationary solutions. Besides the ground state, (1.1) also has an infinite number of nodal solutions which own zero points. (see Berestycki, Lions [1]). The dynamics below and slightly above the ground state is relatively clear in the literature of dispersive PDEs. For NLKG and initial data with energy below the ground state, the dichotomy characterization of blow up v.s. global existence was given by Payne, Sattinger [29] and scattering v.s. blowup below the ground state was obtained by Ibrahim, Masmoudi, Nakanishi [19]. Nakanishi, Schlag [27, 28] obtained the nine set dynamics of the solutions to NLKG with energy slightly above the ground state. In fact they proved the trichotomy forward in time: the solution (i) either blows up at finite time (ii) or globally exists and scatters to zero (iii) or globally exists