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DIFFUSIVE LIMITS OF THE BOLTZMANN EQUATION IN BOUNDED DOMAIN*†

Raffaele Esposito
(*International Research Center M&MOCS, Univ. dell'Aquila, Cisterna di Latina, (LT) 04012 Italy*)
Yan Guo‡
(*Division of Applied Mathematics, Brown University, Providence, RI 02812, U.S.A.*)
Chanwoo Kim
(*Department of Mathematics, University of Wisconsin, Madison, 53706-1325 WI, U.S.A.*)
Rossana Marra
(*Dipartimento di Fisica and Unità INFN, Università di Roma Tor Vergata, 00133 Roma, Italy*)

Abstract

The goal of this paper is to study the important diffusive expansion via an alternative mathematical approach other than that in [21].

Keywords steady and unsteady Boltzmann equation; hydrodynamics limits; boundary effects

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1 Introduction

1.1 Hilbert Expansion with No Boundary Layer Approximations

The hydrodynamic limit of the Boltzmann equation has been the subject of many studies since the pioneering work by Hilbert, who introduced his famous expansion in the Knudsen number ε in [37, 38], realizing the first example of the program he proposed in the sixth of his famous questions [39]. Mathematical results on the closeness of the Hilbert expansion of the Bolzmann equation to the solutions of the compressible Euler equations for small Knudsen number ε, were obtained by

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‡Corresponding author. E-mail: yan_guo@brown.edu

Caflisch [14], and Lachowicz [45], while Nishida [47], Asano and Ukai [4] proved this by different methods.

On a longer time scale ε^{-1}, where diffusion effects become significant, the problem can be faced only in the low Mach numbers regime (Mach number of order ε or smaller) due to the lack of scaling invariance of the compressible Navier-Stokes equations. Hence the Boltzmann solution has been proved to be close to the incompressible Navier-Stokes-Fourier system. Mathematical results were given, among the others, in [11, 18, 31, 33, 34] for smooth solutions. For weak solutions (renormalized solutions), partial results were given, among the others, in [7–10], and the full result for the convergence of the renormalized solutions has been obtained by Golse and Saint-Raymond [27].

Much less is known about the steady solutions. It is worth to notice that, even for fixed Knudsen numbers, the analog of DiPerna-Lions' renormalized solutions [19] is not available for the steady case, due to lack of L^1 and entropy estimates. In [29, 30], steady solutions were constructed in convex domains near Maxwellians, and their positivity was left open. The only other results are for special, essentially one dimensional geometry (see [3] for results at fixed Knudsen numbers and [1, 2, 22, 23] for results at small Knudsen numbers in certain special 2D geometry). In a recent paper [20], via a new L^2-L^∞ framework, we have constructed the steady solution to the Boltzmann equation close to Maxwellians, in 3D general domains, for a gas in contact with a boundary with a prescribed temperature profile modeled by the diffuse reflection boundary condition. The question about positivity of these steady solutions was resolved as a consequence of their dynamical stability. As pointed in [25], despite the importance of steady Navier-Stokes-Fourier equations in applications, it has been an outstanding open problem to derive them from the steady Boltzmann theory.

The goal of our paper is to employ the L^2-L^∞ framework developed in [20] to study the hydrodynamical limit of the solutions to the steady Boltzmann equation, in the low Mach numbers regime, in a general domain with boundary where a temperature profile is specified. We refer to [15, 16, 41–44] for the recent development of L^2-L^∞ framework in various directions.

Let Ω be a bounded open region of $\mathbb{R}^d$ for either $d=2$ or $d=3$. We consider the Boltzmann equation for the distribution density $F(t,x,v)$ with $t\in\mathbb{R}_+:=[0,\infty)$, $x\in\Omega$, $v\in\mathbb{R}^3$. In the diffusive regime, the time evolution of the gas, subject to the action of a field $\vec{G}$, is described by the following *rescaled* Boltzmann equation:

$$\partial_t F+\varepsilon^{-1}v\cdot\nabla_x F+\vec{G}\cdot\nabla_v F=\varepsilon^{-2}Q(F,F), \tag{1.1.1}$$

where the Boltzmann collision operator is defined as