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UNIQUE SOLVABILITY AND DECOMPOSITION METHOD FOR ONE NONLINEAR MULTI-DIMENSIONAL INTEGRO-DIFFERENTIAL PARABOLIC EQUATION

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Abstract. The paper is devoted to the construction and study of the decomposition type semi-discrete scheme for one nonlinear multi-dimensional integro-differential equation of parabolic type. Unique solvability of the first type initial-boundary value problem is given as well. The studied equation is some generalization of integro-differential model, which is based on the well-known Maxwell system arising in mathematical simulation of electromagnetic field penetration into a medium.

Key words. Additive averaged semi-discrete scheme, nonlinear integro-differential multi-dimensional equation, unique solvability.

1. Introduction

The process of electromagnetic field penetration into a medium is described by Maxwell system of nonlinear partial differential equations [16]. In [7], reduction to the integro-differential form of mentioned Maxwell system is proposed

$$\frac{\partial H}{\partial t} = -rot\left[a\left(\int_0^t |rot\,H|^2\,d\tau\right) rot\,H\right], \tag{1}$$

where $H = (H_1, H_2, H_3)$ is a vector of magnetic field and $a = a\,(S)$ is defined for $S \in [0, \infty)$.

One must note that in one-component case $H = (0, 0, U)$ from (1) we get the following integro-differential equation

$$\frac{\partial U}{\partial t} = \nabla\left[a\left(\int_0^t |\nabla U|^2\,d\tau\right) \nabla U\right]. \tag{2}$$

The main characteristic feature of models of type (1) is associated with the appearance of the nonlinear coefficient at the higher derivatives depending on the time integral. This circumstance requires different approach than it is necessary to solve local differential problems. Along with its origin from the applied problems, the studied integro-differential equation (2) may be considered as a natural generalization of the well-known p-Laplacian models (see, for example, [20], [27]). The model (1) for scalar one-dimensional space case was first investigated in [4], [7] and scalar multi-dimensional space case (2) was studied in [5]. Later, these types of integro-differential models were considered in the numerous papers as well (see, for example, [1], [6], [9] - [11], [15], [17] - [19], [26], [29]). The asymptotic behavior and existence of solution by means of Galerkin method for multi-dimensional case and for non-homogeneous right side was studied in [26].

In [4], [5], [7] the solvability of the first boundary value problem is studied using a modified version of the Galerkin method and compactness arguments that are

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used in [20], [27] for investigation of nonlinear elliptic and parabolic models. The uniqueness of the solutions is investigated also in [4], [5], [7]. The asymptotic behavior of solutions is discussed in [6], [10] - [12], [15] and in a number of other works as well. Note also that to numerical resolution of (1) type one-dimensional equations were devoted many works as well (see, for example, [8], [9], [12], [15] and references therein). Many authors study the Rothe scheme, semi-discrete scheme with space variable, finite element and finite difference approximation for a differential and integro-differential models (see, for example, [12], [14] and references therein).

It is very important to study decomposition analogs for the above mentioned multi-dimensional differential and integro-differential models as well. There are some effective algorithms for solving the multi-dimensional problems (see, for example, [2], [3], [21] - [25], [28] and references therein).

Our work is dedicated to the unique solvability of the initial-boundary value problem. We shall focus our attention on (2) multi-dimensional type model Investigations are given in usual Sobolev spaces. Main attention is paid to investigation of semi-discrete additive average scheme. Reduction of multi-dimensional model to one-dimensional ones is proposed and corresponding convergence theorem with rate of convergence is given.

This article is organized as follows. In Section 2 the formulation of the problem and some of its properties are given. Particularly, unique solvability of the stated problem is considered there. Main attention is paid to the construction and investigation of the semi-discrete additive average scheme. This question is discussed in Section 3. Finite difference scheme for one-dimensional case and its implementation are discussed in Sections 4 and 5 respectively. Numerical experiments confirming the theoretical findings are carried out and some of them are given in Section 6. Some conclusions are given in Section 7.

2. Formulation of the problem and unique solvability

Let Ω be a bounded domain in the n-dimensional Euclidean space R^n, with a sufficiently smooth boundary $\partial\Omega$. In the domain $Q = \Omega \times (0,T)$ let us consider the following first type initial-boundary value problem:

$$\frac{\partial U}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[\left(1 + \int_0^t \left|\frac{\partial U}{\partial x_i}\right|^2 d\tau\right)\frac{\partial U}{\partial x_i}\right] = f(x,t), \quad (x,t) \in Q, \tag{3}$$

$$U(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{4}$$

$$U(x,0) = 0, \quad x \in \overline{\Omega}, \tag{5}$$

where $x = (x_1, x_2, ..., x_n)$, T is a fixed positive constant and f is the given function of its arguments.

Theorem 2.1. *If*

$$f, \quad \frac{\partial f}{\partial t}, \quad \sqrt{\psi}\frac{\partial f}{\partial x_i} \in L_2(Q), \quad f(x,0) = 0,$$

where $\psi \in C^\infty(\overline{\Omega})$, $\psi(x) > 0$, for $x \in \Omega$; $\frac{\partial \psi}{\partial \nu} = 0$, for $x \in \partial\Omega$ and ν is the outer normal of $\partial\Omega$, then there exists the unique solution U of problem (3) - (5)