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THE GENERALIZED JACOBIAN OF THE PROJECTION ONTO THE INTERSECTION OF A HALF-SPACE AND A VARIABLE BOX*

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Abstract

This paper is devoted to studying the generalized Jacobian for the projection onto the intersection of a closed half-space and a variable box. This paper derives the explicit formulas of an element in the set of the generalized HS Jacobian for the projection. In particular, we reveal that the generalized HS Jacobian can be formulated as the combination of a diagonal matrix and few rank-one symmetric matrices, which are crucial for future design of efficient second order nonsmooth methods for solving the related optimization problems.

Keywords generalized HS Jacobian; projection; intersection of a half-space and a variable box

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1 Introduction

Given $(x,t) \in \Re^n \times \Re$ and $r > 0$, we consider the following optimization problem:

$$\begin{array}{ll} \min\limits_{y,\tau} & \frac{1}{2}\|y-x\|^2 + \frac{1}{2}(\tau - t)^2 \\ \text{s.t.} & e_n^{\mathrm{T}} y \le r\tau, \\ & 0 \le y \le \tau e_n, \end{array} \tag{P}$$

where e_n denotes the vector of all ones in $\Re^n$. For simplicity, we define $\mathcal{B}_r$ as the intersection of a closed half-space and a variable box:

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$$\mathcal{B}_r := \{(y,\tau) \in \Re^n \times \Re \mid e_n^{\mathrm{T}} y \le r\tau, 0 \le y \le \tau e_n\}. \tag{1}$$

It is obvious (cf. [13]) that $\mathcal{B}_r$ is a polyhedral convex set. It is also easy to see that the optimal solution to (P) is the projection $\Pi_{\mathcal{B}_r}(x,t)$ of (x,t) onto $\mathcal{B}_r$.

Fast solvers for computing the projection $\Pi_{\mathcal{B}_r}(\cdot,\cdot)$ and the explicit formulas of the generalized Jacobian for $\Pi_{\mathcal{B}_r}(\cdot,\cdot)$ are needed in designing efficient algorithms such as augmented Lagrangian methods for the optimization problems subject to the epigraph of the vector k-norm functions and the matrix Ky Fan k-norm functions. It is worth mentioning that Ky Fan k-norm functions appear frequently in matrix optimization problems. For instance, the problem of finding the fastest Markov chain on a graph which can be recast as minimizing the Ky Fan 2-norm was studied in [1, 2]. Besides, the structured low rank matrix approximation [3] arising in many areas is a kind of matrix optimization problem involving Ky Fan k-norm. For more applications, we refer the reader to [4, 14] and references therein.

Note that the explicit formulas of the projection $\Pi_{\mathcal{B}_r}(\cdot,\cdot)$ and the fast algorithm proposed for finding the projection $\Pi_{\mathcal{B}_r}(\cdot,\cdot)$ were studied in [12], we mainly focus on the computation of the generalized Jacobian of $\Pi_{\mathcal{B}_r}(\cdot,\cdot)$ in this paper. The generalized Jacobian of the projector $\Pi_{\mathcal{B}_r}(\cdot,\cdot)$ plays an essential role in the algorithmic design of the second order nonsmooth methods for solving optimization problems with constraints involving $\mathcal{B}_r$. The efficiency and robustness of the algorithms which utilize the second order information like the semismooth Newton augmented Lagrangian methods (SSNAL) for solving the optimization problems have been demonstrated in many works [8–11, 15]. Thus, the efficient computation of generalized Jacobian of the metric projector $\Pi_{\mathcal{B}_r}(\cdot,\cdot)$ deserves our research efforts.

As far as we know, the generalized Jacobian of the projection $\Pi_{\mathcal{B}_r}(\cdot,\cdot)$ has not been studied before. The main contribution of this paper is to study the explicit formulas of the generalized HS-Jacobian [7,10] for the projection $\Pi_{\mathcal{B}_r}(\cdot,\cdot)$. Due to the difficulty of characterizing the B-subdifferential and the Clarke generalized Jacobian of the projection onto the nonempty polyhedral convex set, Han and Sun [7] proposed a particular multi-valued mapping (HS-Jacobian) as an alternative for the generalized Jacobian of the projector onto polyhedral sets. Recently, the authors [10] derived an explicit formula for constructing the generalized HS-Jacobian and thus the computation of the generalized Jacobian was further simplified. Based on their works, we are able to study the generalized HS-Jacobian of $\Pi_{\mathcal{B}_r}(\cdot,\cdot)$ in an efficient way.

The remaining parts of this paper are organized as follows. Section 2 reviews some preliminary results on the generalized Jacobian of the projection onto the general polyhedral convex set. This lays the foundation for the study on the generalized HS-Jacobian of the projection $\Pi_{\mathcal{B}_r}(\cdot,\cdot)$ in Section 3. Finally, we conclude the paper