J. Math. Study
doi: 10.4208/jms.v54n1.21.04

Vol. **54**, No. 1, pp. 81-88
March 2021

Remarks on a Mean Field Equation on $\mathbb{S}^2$

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Received May 27, 2019; Accepted September 23, 2019;
Published online January 12, 2021.

Dedicated to Professors Sun-Yung Alice Chang and Paul C. Yang on their 70th birthdays

Abstract. In this note, we study symmetry of solutions of the elliptic equation

$$-\Delta_{\mathbb{S}^2} u + 3 = e^{2u} \quad \text{on} \quad \mathbb{S}^2,$$

that arises in the consideration of rigidity problem of Hawking mass in general relativity. We provide various conditions under which this equation has only constant solutions, and consequently imply the rigidity of Hawking mass for stable constant mean curvature (CMC) sphere.

AMS subject classifications: 35J61, 58J05, 35B06

Key words: Semilinear elliptic equation, sphere covering inequality, rigidity of Hawking mass.

1 Introduction

The main aim of this note is to study the semilinear elliptic equation

$$-\Delta_{\mathbb{S}^2} u + \alpha = e^{2u} \tag{1.1}$$

on the standard $\mathbb{S}^2$. Here u is a smooth function on $\mathbb{S}^2$ and α is a positive constant.

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When $\alpha=1$, (1.1) means that the conformal metric $e^{2u}g_{\mathbb{S}^2}$ has constant curvature 1. Therefore all solutions are given by the pull back of the standard metric by Mobius transformations. This and more general statements also follow from the powerful method of moving plane (see [5, 9]). The latter approach can be used to show (1.1) has only constant solution when $0<\alpha<1$ (see [16]). More recently, the sphere covering inequality was discovered in [12] and applied to show all solutions to equation (1.1) must be constant functions for $1<\alpha\le 2$. In particular, this confirms a long-standing conjecture of Chang-Yang ([3, 4]) concerning the best constant in Moser-Trudinger type inequalities. Sphere covering inequality and its generalization can also be used to solve many uniqueness and symmetry problems from mathematical physics (see [1, 10, 12] and many references therein). [10] explains the sphere covering inequality from the point view of comparison geometry and provides some further generalizations. In contrast, for $2<\alpha<3$, nontrivial axially symmetric solutions were found in [15]. The multiplicity of these nontrivial axially symmetric solutions was carefully discussed in [8]. More recently, non-axially symmetric solutions to (1.1) for $\alpha>4$ but close to 4 were found in [11]. In related developments, topological degree of (1.1) for $\alpha\notin\mathbb{Z}$ was computed in [6, 14, 15]. We refer the readers to the survey article [20] for more details of mean field equations on a closed surface.

Recently, [18, 19] discovered the interesting connection between the equation (1.1) with $\alpha=3$ and rigidity problems involving Hawking mass in general relativity. Among other results, it was shown in [18] that for $2<\alpha<4$, any even solution to (1.1) must be axially symmetric. In particular, when $\alpha=3$, any even solution, $u(x)=u(-x)$ for all $x\in\mathbb{S}^2$, must be a constant function. It is also conjectured in [18, Section 3] that for $2<\alpha\le 3$, any solution to (1.1) must be axially symmetric. Our note is motivated by this conjecture. Our main result is

Theorem 1.1. *Assume $2<\alpha\le 3$ and $u\in C^\infty(\mathbb{S}^2)$ is a solution to*

$$-\Delta_{\mathbb{S}^2}u+\alpha=e^{2u}.$$

If for some $p\in\mathbb{S}^2$, $\nabla u(p)=0$ and $D^2u(p)$ has two equal eigenvalues, then u is axially symmetric with respect to p. In particular, in the case $\alpha=3$, u must be a constant function.

We may call the point p in the assumption as an umbilical critical point of u. So the theorem reads as: for $2<\alpha\le 3$, any solution with an umbilical critical point must be axially symmetric with respect to that point. Here we do not know whether the solution is even or not. On the other hand, the approach to Theorem 1.1 can help us relax the even assumption in [18] a little bit. One typical example is

Theorem 1.2. *Assume $2<\alpha\le 3$ and $u\in C^\infty(\mathbb{S}^2)$ is a solution to*

$$-\Delta_{\mathbb{S}^2}u+\alpha=e^{2u}.$$

If every large circle splits $\mathbb{S}^2$ as two half sphere with equal area under the metric $e^{2u}g_{\mathbb{S}^2}$, then u is axially symmetric with respect to some point. In particular, in the case $\alpha=3$, u must be a constant function.