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Initial and Boundary Value Problem for a System of Balance Laws from Chemotaxis: Global Dynamics and Diffusivity Limit

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Abstract. In this paper, we study long-time dynamics and diffusion limit of large-data solutions to a system of balance laws arising from a chemotaxis model with logarithmic sensitivity and nonlinear production/degradation rate. Utilizing energy methods, we show that under time-dependent Dirichlet boundary conditions, long-time dynamics of solutions are driven by their boundary data, and there is no restriction on the magnitude of initial energy. Moreover, the zero chemical diffusivity limit is established under zero Dirichlet boundary conditions, which has not been observed in previous studies on related models.

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1 Introduction

1.1 Overview

This paper is oriented around the initial-boundary value problem:

$$\begin{cases} p_t-(pq)_x=p_{xx}, & x\in(0,1), \quad t>0,\\ q_t-(p^\gamma+\varepsilon q^2)_x=\varepsilon q_{xx}, & x\in(0,1), \quad t>0,\\ (p,q)(x,0)=(p_0,q_0)(x), & x\in(0,1),\\ p|_{x=0,1}=\alpha(t), \quad q|_{x=0,1}=\beta(t), & t>0, \quad \text{when} \ \ \varepsilon>0,\\ p|_{x=0,1}=\alpha(t), & t>0, \quad \text{when} \ \ \varepsilon=0, \end{cases} \tag{1.1}$$

where $\gamma>1$ and ε are constant parameters, and $p_0(x)$, $q_0(x)$, $\alpha(t)$, and $\beta(t)$ are given functions. Assuming appropriate conditions for $p_0(x)$, $q_0(x)$, $\alpha(t)$ and $\beta(t)$, we establish global stability of strong solutions to (1.1). Moreover, when $\alpha(t)=\beta(t)\equiv 0$, we show that solutions to (1.1) with $\varepsilon>0$ converge to that with $\varepsilon=0$, as $\varepsilon\to 0$, in certain topology.

1.2 Background

The system of balance laws in problem (1.1) is derived from the following chemotaxis model of Keller-Segel type with logarithmic sensitivity:

$$\begin{cases} u_t=Du_{xx}-\chi(u(\ln c)_x)_x,\\ c_t=\varepsilon c_{xx}-\mu\psi(u)c-\sigma c, \end{cases} \tag{1.2}$$

when $\psi(u)=u^\gamma$. Here, the unknown functions $u(x,t)$ and $c(x,t)$ denote, respectively, density of cellular population and concentration of chemical signal at position x at time t. The parameters: $D>0$ stands for diffusion coefficient of cellular density, $\chi\neq 0$ coefficient of chemotactic sensitivity, $\varepsilon\geq 0$ diffusion coefficient of chemical signal, $\mu\psi(u)\neq 0$ density-dependent production/degradation rate of chemical signal, $\psi(u)>0$ given function of cellular density, and $\sigma>0$ denotes natural degradation rate of chemical signal. The sign of χ dictates whether the chemotaxis is attractive ($\chi>0$) or repulsive ($\chi<0$), and $|\chi|$ measures the strength of chemotactic response. Moreover, the logarithmic sensitivity entails that the chemotactic response of cellular population to chemical signal follows Weber-Fechner's law (c.f. [1, 2, 4, 14]), which appeared in one of the original Keller-Segel models of chemotaxis (c.f. [15]).

Derivation of the system of balance laws in (1.1) from (1.2) can be realized by