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A Weak Galerkin Finite Element Method for p-Laplacian Problem

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Abstract. In this paper, we introduce a weak Galerkin (WG) finite element method for p-Laplacian problem on general polytopal mesh. The quasi-optimal error estimates of the weak Galerkin finite element approximation are obtained. The numerical examples confirm the theory.

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Key words: Weak Galerkin, finite element methods, p-Laplacian, polyhedral meshes.

1. Introduction

We consider the following p-Laplacian problem

$$\begin{aligned} \nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right)&=f \quad \text{in} \quad \Omega,\\ u&=0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

where $1<p<\infty$.

The p-Laplacian problem has many applications including filtration, power-law materials and quasi-Newtonian flows. Finite element analysis of the p-Laplacian has been extensively studied in the literature. The quasi-norm approach introduced in [2] provides sharper error bounds for finite element solutions of the p-Laplacian problems. The quasi-norm error estimates have been derived for different finite element approximations in [4, 8, 9].

The weak Galerkin finite element method is an effective and flexible numerical technique for solving partial differential equations. It is a natural extension of the standard Galerkin finite element method where classical derivatives are substituted by weakly defined derivatives on functions with discontinuities. The WG method was first introduced in [15, 16] and then has been applied to solve various PDEs such as second order elliptic

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equations, biharmonic equations, Stokes equations, convection dominant problems, hyperbolic equations, and Maxwell's equations [1,3,5,6,11–14,17–24].

In this paper, we introduce a WG finite element method for solving the p-Laplacian problem. Error estimates are obtained in different norms. The numerical examples tested on hybrid polygonal meshes confirm the theoretical findings.

2. Finite Element Methods

For any given polygon $D \subseteq \Omega$, we use the standard definition of Sobolev spaces $H^s(D)$ with $s \geq 0$. The associated inner product, norm, and semi-norm in $H^s(D)$ are denoted by $(\cdot,\cdot)_{s,D}$, $\|\cdot\|_{s,D}$, and $|\cdot|_{s,D}, 0 \leq s$, respectively. When $s = 0$, $H^0(D)$ coincides with the space of square integrable functions $L^2(D)$. In this case, the subscript s is suppressed from the notation of norm, semi-norm, and inner products. Furthermore, for $D = \Omega$ the subscript D is also suppressed.

Let $\mathscr{T}_h$ be a partition of a domain Ω consisting of polygons in two dimension or polyhedra in three dimension satisfying a set of conditions specified in [16]. Denote by $\mathscr{E}_h$ the set of all edges or flat faces in $\mathscr{T}_h$, and let $\mathscr{E}_h^0 = \mathscr{E}_h \backslash \partial\Omega$ be the set of all interior edges or flat faces. For every element $T \in \mathscr{T}_h$, we denote by h_T its diameter and mesh size $h = \max_{T\in\mathscr{T}_h} h_T$ for $\mathscr{T}_h$.

For $k \geq 1$, we define the finite element spaces

$$
\begin{aligned}
V_h &:= \big\{v = \{v_0, v_b\} : v|_T \in P_k(T) \times P_k(e), e \in \partial T,\ T \in \mathscr{T}_h\big\},\\
V_h^0 &:= \big\{v \in V_h : v_b = 0 \text{ on } \partial\Omega\big\}.
\end{aligned}
$$

For any $v = \{v_0, v_b\}$, the weak gradient $\nabla_w v \in [P_{k-1}(T)]^d$ is defined on T by

$$
(\nabla_w v,\ \varphi)_T := (v_0, \nabla\cdot\varphi)_T - \langle v_b, \varphi\cdot\boldsymbol{n}\rangle_{\partial T} \quad \text{for all} \quad \varphi \in [P_{k-1}(T)]^d. \tag{2.1}
$$

We introduce also the bilinear forms

$$
s(v,w) := \sum_{T\in\mathscr{T}_h} h_T^{-1}\langle v_0 - v_b, w_0 - w_b\rangle_{\partial T}, \tag{2.2}
$$

$$
a(v,w) := \sum_{T\in\mathscr{T}_h} \big(|\nabla_w v|^{p-2}\nabla_w v, \nabla_w w\big)_T + s(v,w). \tag{2.3}
$$

Let Q_0, Q_b and $\mathbb{Q}_h$ be the locally defined L^2 projections onto $P_k(T), P_k(e)$ and $[P_{k-1}]^d$ accordingly on each element $T \in \mathscr{T}_h$ and $e \in \partial T$. For the true solution u of (1.1), we define $Q_h u$ as

$$
Q_h u := \{Q_0 u, Q_b u\} \in V_h^0.
$$

Algorithm 2.1. A numerical approximation for (1.1) can be obtained by seeking $u_h = \{u_0, u_b\} \in V_h^0$ satisfying the following equation:

$$
a(u_h, v) = (f, v_0) \quad \text{for all} \quad v = \{v_0, v_b\} \in V_h^0. \tag{2.4}
$$