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A Restricted Linearised Augmented Lagrangian Method for Euler's Elastica Model

Yinghui Zhang, Xiaojuan Deng, Xing Zhao and Hongwei Li*

School of Mathematical Sciences and Beijing Advanced Innovation Center for Imaging Theory and Technology, Capital Normal University, Beijing 100048, China.

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Abstract. A simple cutting-off strategy for the augmented Lagrangian formulation for minimising the Euler's elastica energy is introduced. It is connected to a discovered internal inconsistency of the model and helps to decouple the tricky dependence between auxiliary splitting variables, thus fixing the problem mentioned. Numerical experiments show that the method converges much faster than conventional algorithms, provides a better parameter-tuning and ensures the higher quality of image restorations.

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Key words: Euler's elastica, augmented Lagrangian, image denoising.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^2$. The classical image denoising problem consists in deriving the clean image u from a given noisy image $f : \Omega \to \mathbb{R}$,

$$f = u + n,$$

where n denotes the noise. One of the most popular and successful variational methods for solving this problem — viz. ROF, developed by Rudin *et al.* [25], is formulated as the following minimisation problem:

$$\min_u \int_\Omega |\nabla u| + \frac{\lambda}{2} \int_\Omega (f-u)^2,$$

where $\lambda > 0$ is a scalar weighting parameter. Due to total variation term in the energy functional, the ROF model effectively preserves edges during noise removal. Several fast algorithms have been proposed for solving the ROF model [12, 24, 29, 31].

*Corresponding author. *Email addresses:* zhangyinghui9@cnu.edu.cn (Y.-H. Zhang), hongwei.li91@cnu.edu.cn (H.-W. Li), xiaojuan_deng@cnu.edu.cn (X.-J. Deng), zhaoxing_1999@126.com (X. Zhao)

Despite its appealing properties, the ROF model yields the staircase effect and fails to preserve image contrasts. To overcome these drawbacks, some higher order variational models have been proposed [4,6,7,19–21,39]. Among them, Euler's elastica energy functional has a wide application in image processing, such as image inpainting [5,26,33], image segmentation [3,11,16,27,40], image restoration [2,13,30,32,39], image decomposition [18] and image reconstruction [34,35]. For image restoration, the Euler's elastica model is defined as

$$\min_u \int_\Omega \left(a + b\kappa^2\right) |\nabla u| d\Omega + \frac{\lambda}{s} \int_\Gamma \|Au - f\|^s d\Omega, \tag{1.1}$$

where a, b are weighting parameters and $\kappa = \nabla \cdot (\nabla u / |\nabla u|)$ denotes the curvature. The Euler's elastica functional was introduced by Mumford *et al.* [22] for segmenting images with occlusions. Various conventional algorithms employed in solving the Euler's elastica model [5,17,23,26] require effective discretisation schemes for solving the Euler-Lagrange equation. However, because of nonlinearity and nonconvexity of the curvature term, the corresponding computations are of high complexity and time-consuming.

Several fast numerical algorithms [2,8,9,11,15] proposed recently employ the augmented Lagrangian and other techniques, such as primal-dual operator splitting or linearisation to convert the original problem into simpler subproblems. Tai *et al.* [28] reformulated (1.1) as a constrained optimisation problem by using the idea of augmented Lagrangian. Consequently, an alternating minimisation method can be used to transform the original problem into a series of subproblems the analytic solution of which can be obtained. Zhang and Chen [37] exploited an augmented Lagrangian-based primal dual approach in order to reduce the number of auxiliary variables. Yashtini and Kang [33] suggested a simpler method with a fewer number of parameters and solved the corresponding subproblems by FFT and shrinkage operator. Besides, in order to develop fast algorithms, they considered the Euler's elastica energy minimisation problem as a weighted ROF problem. On the other hand, the application of FFT is limited to periodic boundary conditions and to overcome this constrain, Tai [14] replaced FFT by an inexpensive arithmetic operation. Following similar ideas, Duan *et al.* [10] decoupled curvature term by introducing an additional variable and replaced the FFT by Gauss-Seidel iterations, thus reducing computational cost. Consequently, Zhang *et al.* [36,38] used the linearisation of quadratic terms in the Euler-Lagrange equation, so that one can find the exact solutions of the corresponding subproblems.

Nevertheless, because of the mutual dependence of auxiliary variables, all known fast solvers for the Euler's elastica model (1.1) may suffer during the parameter tuning. Therefore, in this work we develop a cutting-off strategy, which provides an easier parameter tuning, a higher quality of restoration and a faster convergence of augmented Lagrangian based algorithm for (1.1). It is based on a key observation of inconsistency of the conventional augmented Lagrangian approach for (1.1) — viz. if b tends to 0, the Euler's elastica model (1.1) approaches the following ROF model:

$$\min_u \int_\Omega a |\nabla u| d\Omega + \frac{\lambda}{2} \int_\Omega (u - f)^2 d\Omega. \tag{1.2}$$