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THE ARBITRARY LAGRANGIAN-EULERIAN FINITE ELEMENT METHOD FOR A TRANSIENT STOKES/PARABOLIC INTERFACE PROBLEM

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Abstract. In this paper, a type of nonconservative arbitrary Lagrangian-Eulerian (ALE) finite element method is developed and analyzed in the monolithic frame for a transient Stokes/parabolic moving interface problem with jump coefficients. The mixed and the standard finite element approximations are adopted for the transient Stokes equations and the parabolic equation on either side of the moving interface, respectively. The stability and optimal convergence properties of both semi- and full discretizations are analyzed in terms of the energy norm. The developed numerical method can be generally extended to the realistic fluid-structure interaction (FSI) problems in a time-dependent domain with a moving interface.

Key words. Arbitrary Lagrangian-Eulerian (ALE) method, mixed finite element method (FEM), fluid-structure interactions (FSI), Stokes/parabolic interface problem, stability, optimal convergence.

1. Introduction

In this paper, we study the following coupled system of partial differential equations (PDEs), which consists of the transient Stokes equations and a parabolic equation defined in respective time-dependent subdomains and separated by a moving interface:

$$
(1)\qquad \begin{cases}
\frac{\partial \boldsymbol{v}_1}{\partial t} - \nabla\cdot(\mu_1\nabla\boldsymbol{v}_1) + \nabla p_1 &= \boldsymbol{f}_1, & \text{in } \Omega_t^1\times\mathcal{I}\\
\nabla\cdot\boldsymbol{v}_1 &= 0, & \text{in } \Omega_t^1\times\mathcal{I}\\
\boldsymbol{v}_1 &= 0, & \text{on } \partial\Omega_t^1\backslash\Gamma_t\times\mathcal{I}\\
\boldsymbol{v}_1(\boldsymbol{x},0) &= \boldsymbol{v}_1^0, & \text{in } \hat{\Omega}^1=\Omega_0^1\\
\frac{\partial \boldsymbol{v}_2}{\partial t} - \nabla\cdot(\mu_2\nabla\boldsymbol{v}_2) &= \boldsymbol{f}_2, & \text{in } \Omega_t^2\times\mathcal{I}\\
\boldsymbol{v}_2 &= 0, & \text{on } \partial\Omega_t^2\backslash\Gamma_t\times\mathcal{I}\\
\boldsymbol{v}_2(\boldsymbol{x},0) &= \boldsymbol{v}_2^0, & \text{in } \hat{\Omega}^2=\Omega_0^2\\
\boldsymbol{v}_1 &= \boldsymbol{v}_2, & \text{on } \Gamma_t\times\mathcal{I}\\
(-p_1\boldsymbol{I}+\mu_1\nabla\boldsymbol{v}_1)\,\boldsymbol{n}_1+\mu_2\nabla\boldsymbol{v}_2\boldsymbol{n}_2 &= \boldsymbol{\tau}, & \text{on } \Gamma_t\times\mathcal{I}
\end{cases}
$$

where $\Omega\subset\mathbb{R}^d$ $(d=2,3)$, $\mathcal{I}=(0,T]$ $(T>0)$, and two subdomains, $\Omega_t^i:=\Omega_i(t)\subset\Omega$ $(i=1,2)$ $(0\le t\le T)$, satisfy $\overline{\Omega_t^1}\cup\overline{\Omega_t^2}=\overline{\Omega}$, $\Omega_t^1\cap\Omega_t^2=\emptyset$ and are separated by a moving interface: $\Gamma_t:=\Gamma(t)=\partial\Omega_t^1\cap\partial\Omega_t^2$. Γ_t may move/deform along with $t\in\mathcal{I}$, then may cause Ω_t^i $(i=1,2)$ to change with $t\in\mathcal{I}$ as well, which are thus termed as the current (Eulerian) domains with respect to $\boldsymbol{x}_i$ in contrast to their initial (reference/Lagrangian) domains, $\hat{\Omega}^i:=\Omega_0^i$ with respect to $\hat{\boldsymbol{x}}_i$ $(i=1,2)$, where, a *flow map* is defined from $\hat{\Omega}^i$ to Ω_t^i, as: $\hat{\boldsymbol{x}}_i\mapsto\boldsymbol{x}_i(\hat{\boldsymbol{x}}_i,t)$ such that $\boldsymbol{x}_i(\hat{\boldsymbol{x}}_i,t)=\hat{\boldsymbol{x}}_i+\hat{\boldsymbol{u}}_i(\hat{\boldsymbol{x}}_i,t),\forall t\in\mathcal{I}$, where $\hat{\boldsymbol{u}}_i$ $(i=1,2)$ is the displacement field in the Lagrangian frame. In addition, μ_1 and μ_2 are jump constants. In what follows, we set $\hat{\psi}_i=\hat{\psi}_i(\hat{\boldsymbol{x}}_i,t)$ which equals $\psi_i(\boldsymbol{x}_i(\hat{\boldsymbol{x}}_i,t),t)$ $(i=1,2)$. Correspondingly,

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the deformation gradient tensor is defined as $\mathbf{F}^i := \nabla_{\hat{\boldsymbol{x}}_i} \boldsymbol{x}_i = \boldsymbol{I} + \nabla_{\hat{\boldsymbol{x}}_i} \hat{\boldsymbol{u}}_i$, and $\boldsymbol{J}^i = \det(\mathbf{F}^i)$ $(i = 1, 2)$.

The model problem (1) can be essentially considered as a linearized fluid-structure interaction (FSI) problem [5, 9, 16, 1, 2, 14, 17], where the transient Stokes equations describe the fluid motion in terms of the fluid velocity $\boldsymbol{v}_1$ and pressure p_1, and the parabolic equation just stands for a dynamic linear elasticity problem in terms of the structural velocity $\boldsymbol{v}_2$ [19]. In addition, μ_1 can represent the fluid viscosity, and μ_2 denotes the elastic parameter of the structure. Hence, (1) holds the essential characteristic of FSI problems at least partially, that is, two different types of time-dependent governing equations bearing with different primary unknowns and different compressibility/constitutive relations are defined on either side of the moving interface. FSI problems describe the coupled dynamics of fluid mechanics and structure mechanics through the moving interface. They are classical multiphysics problems and as such, have a diverse range of applications in engineering. A key factor in the simulation of such problems comes from the deformation of the domains due to the evolving fluid flow acting on the surface of structure and thus making the structure deformable. Specifically, we are looking at a two-way coupled system in which the fluid flow affects the structural deformation, at the same time, the motion of the structure impacts the fluid flow through their interfaces. The thing that every FSI problem has in common is that the subdomains in which the coupled system is defined will move with respect to time due to the interface motion, that is, the subdomains are no longer fixed. The movement of the domain/interface can be in the form of a rotation, translation and/or deformation.

In order to take the domain motion into consideration, the arbitrary Lagrangian-Eulerian (ALE) technique is always adopted to redescribe the moving boundary/interface problem, and then a conservative ALE-finite volume/element method [8, 13] is usually developed to discretize the corresponding moving boundary/interface problems in order to account for the preservation of geometric conservation law (GCL) [7]. In the case of finite element spatial discretizations, the relationships between GCL condition, stability and accuracy properties of the numerical scheme, have not been completely clarified yet [14]. Recently, GCL condition is proved to be neither necessary nor sufficient for the stability of ALE-finite element scheme [4]. Then, a nonconservative type of ALE-finite element discretization, which does not need to preserve the GCL condition, becomes promising due to its relatively simpler implementation and less storage since only one-level mesh is involved in the nonconservative ALE-finite element method [19], in contrast with the conservative ALE method in which two-level meshes must be employed.

Towards an effective and practical ALE-finite element approximation to a realistic also complicated FSI problem, in this paper we will start with a simplified FSI model – a transient Stokes/parabolic moving interface problem, develop its nonconservative ALE-finite element approximation in semi- and fully discrete schemes, and analyze their stability and optimal convergence properties. Afterwards, our method will be more likely extended to a realistic FSI problem that was first studied in [11, 12] where however a more complicated conservative scheme of ALE method is adopted, and, our simpler nonconservative ALE scheme will be still stable as well as possess an optimal error estimate for FSI problems, which will be studied in our next paper.

The structure of this paper is organized as follows: in Section 2 we introduce the ALE mapping as well as define the nonconservative weak form of the presented Stokes/parabolic interface problem. Then we define the semi-discrete ALE finite