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A STABILIZER FREE WEAK GALERKIN FINITE ELEMENT METHOD FOR GENERAL SECOND-ORDER ELLIPTIC PROBLEM

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Abstract. This paper proposes a stabilizer free weak Galerkin (SFWG) finite element method for the convection-diffusion-reaction equation in the diffusion-dominated regime. The object of using the SFWG method is to obtain a simple formulation which makes the SFWG algorithm (9) more efficient and the numerical programming easier. The optimal rates of convergence of numerical errors of $\mathcal{O}(h^k)$ in H^1 and $\mathcal{O}(h^{k+1})$ in L^2 norms are achieved under conditions $\left(P_k(K), P_k(e), [P_j(K)]^2\right), j = k+1, k = 1, 2$ finite element spaces. Numerical experiments are reported to verify the accuracy and efficiency of the SFWG method.

Key words. Stabilizer free weak Galerkin methods, weak Galerkin finite element methods, weak gradient, error estimates.

1. Introduction

In this paper, we are concerned with the development of numerical methods for the following partial differential equation with boundary conditions using a stabilizer free weak Galerkin finite element method

$$-\nabla\cdot(\alpha\nabla u) + \boldsymbol{\beta}\cdot\nabla u + cu = f \quad \text{in } \Omega, \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where Ω is a polygonal or polyhedra domain in $\mathbb{R}^d (d = 2, 3)$, $\alpha = \alpha(x)$ is the diffusion coefficient matrix, $\boldsymbol{\beta} = \boldsymbol{\beta}(x)$ is the convection coefficient and $c = c(x)$ is the reaction coefficient in relevant applications. We suppose that $\alpha = (\alpha_{ij}(x))_{d\times d} \in [W^{1,\infty}(\Omega)]^{d\times d}, 0 \le c(x) \le M, \boldsymbol{\beta} \in [W^{1,\infty}(\Omega)]^d$ and $c - \frac{1}{2}\nabla\cdot\boldsymbol{\beta} > c_0 > 0$ for some constant c_0 and there exists positive constants $\alpha_m \le \alpha_M$ such that

$$\alpha_m \xi^T\xi \le \xi^T\alpha(x)\xi \le \alpha_M\xi^T\xi, \quad \forall\xi\in\mathbb{R}^d, x\in\Omega.$$

The convection-diffusion equation has numerous practical applications in many fields such as materials sciences, fluid flows, and image processing. There are several numerical methods in existing literature for solving the convection-diffusion equation.

The weak form of the problem (1)-(2) is to find $u \in H_0^1(\Omega)$ such that

$$(\alpha\nabla u, \nabla v) + (\boldsymbol{\beta}\cdot\nabla u, v) + (cu, v) = (f, v), \forall v \in H_0^1(\Omega). \tag{3}$$

The standard weak Galerkin method for the problem (1)-(2) seeks weak Galerkin finite element approximation $u_h = \{u_0, u_b\}$ satisfying

$$(\alpha\nabla_w u, \nabla_w v) + (\boldsymbol{\beta}\cdot\nabla_w u, v) + (cu, v) + s(u_h, v) = (f, v), \tag{4}$$

for all $v = \{v_0, v_b\}$ satisfying $v_b = 0$ on $\partial\Omega$, where ∇_w is the weak gradient operator and $s(u_h, v)$ in (4) is a stabilizer term that ensures a sufficient weak continuity for the numerical approximating. Recently, the weak Galerkin method has been developed to solve the elliptic equations [3, 6, 5], singularly perturbed reaction-diffusion

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problems [1], the biharmonic problems [9], the Helmholtz equation [8], and the Maxwell equations [7]. More recently, Lin, et al. in [4], proposed a simple WG method for the convection-diffusion-reaction problem (1)-(2) with singular perturbation. One of the complexities of the WG methods and other discontinuous finite element methods is contained the stabilization terms. To reduce the programming complexity, the stabilizer free weak Galerkin finite element method, introduced by Ye and Zhang in [13], refers to the numerical techniques for solving Poisson equation on polytopal meshes in 2D or 3D, where there is a $j_0 > 0$ so that as long as the degree j of the weak gradient satisfies $j \geq j_0$, the new scheme will work and the optimal order of convergence can be achieved. In [2], Al-Taweel and Wang proved the optimal degree of weak gradient of the SFWG method to improve the efficiency of SFWG and to avoid the numerical difficulties associated with using high degree weak gradients. The benefits of using the SFWG method compared to the standard weak Galerkin method (4) are twofold: firstly, the SFWG method has a simple formulation which is closer to the weak form (3) and thus the implementation of the SFWG finite element method is easier than that of the standard weak Galerkin method; secondly and more importantly, it is more efficient than the standard WG method (4). The goal of this article is to study a stabilizer free weak Galerkin finite element method for solving convection-diffusion-reaction equations (1)-(2) on uniform triangular partitions and then establish the error analysis in the H^1 norm and L^2 norm.

This paper is organized as follows: In Section 2, we define weak gradient, weak divergence, and describe our SFWG finite element spaces and the SFWG scheme for the convection-diffusion-reaction equations (1)-(2). In Section 3, we will derive optimal order L^2 error estimates for the SFWG finite element method for solving the equations (1)-(2). Numerical experiment results are presented in Section 4 to validate the theoretical results. Finally, in Section 5, we present some concluding remarks.

2. Weak Galerkin Finite Element Schemes

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of convex polygons in 2D or polyhedra in 3D. Suppose that $\mathcal{T}_h$ is shape regular in the sense defined by (11)-(12). Let $\mathcal{E}_h$ be the set of all edges in $\mathcal{T}_h$, let let $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$ be the set of all interior edges. For each element $K \in \mathcal{T}_h$, denote by h_K the diameter of K, and $h = max_{K\in\mathcal{T}_h} h_K$ the mesh size of $\mathcal{T}_h$.

On each K, let $P_k(K)$ be the space of all polynomials with degree k or less. Let V_h be the weak Galerkin finite element space associated with $K \in \mathcal{T}_h$ defined as follows:

$$V_h = \{v = \{v_0, v_b\} : v_0|_K \in P_k(K), v_b|_e \in P_k(e), K \in \mathcal{T}_h, e \in \partial K\}, \tag{5}$$

where $k \geq 1$ is a given integer. In this instance, the component v_0 symbolizes the interior value of v, and the component v_b symbolizes the edge value of v on each K and e, respectively. Let V_h^0 be the subspace of V_h defined as:

$$V_h^0 = \{v : v \in V_h, v_b = 0 \text{ on } \partial\Omega\}. \tag{6}$$

Definition 2.1. *(Weak Gradient) For any $v = \{v_0, v_b\}$, the weak gradient $\nabla_w v \in [P_j(K)]^d$, where $j > k$, is defined on K as the unique polynomial satisfying*

$$(\nabla_w v, \mathbf{q})_K = -(v_0, \nabla\cdot\mathbf{q})_K + \langle v_b, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial K}, \quad \forall \mathbf{q} \in [P_j(K)]^d, \tag{7}$$

where $\mathbf{n}$ is the unit outward normal vector of ∂K.