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On a Special Generalized Mixture Class of Probabilistic Models

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Abstract In this paper, we develop a new mathematical strategy to create flexible lifetime distributions. This strategy is based on a special generalized mixture derived to the one involved in the so-called weighted exponential distribution. Thus, we introduce a new class of lifetime distributions called "special generalized mixture" class and discussed its qualities. In particular, a short list of new lifetime distributions is presented in details, with a focus on the one based on the Lomax distribution. Different mathematical properties are described, including distributional results, diverse moments measures, incomplete moments, characteristic function and bivariate extensions. Then, the applicability of the new class is investigated through the model parameters based on the Lomax distribution and the analysis of a practical data set.

Keywords Lifetime distribution, Weighted exponential distribution, Moments, Copula, Data analysis.

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1. Introduction

Despite certain qualities, the exponential distribution suffers from a lack of flexibility to model a large panel of lifetime phenomena. In order to improve it on this aspect, several extensions adding a shape parameter have been proposed, beginning with the notorious Weibull distribution. In particular, an alternative was proposed by the weighted exponential (WE) distribution introduced by [11]. It is based on the idea of [2], defining the related probability density function (PDF) by

$$f(x;\alpha,\lambda)=\frac{1}{P(\alpha X_1>X_2)}f_*(x;\lambda)F_*(\alpha x;\lambda),\quad x>0,$$

(and zero otherwise), where $\alpha>0$, and X_1 and X_2 are two independent and identically distributed random variables defined on a certain probability space, say $(\Omega,\mathcal{A},P)$, with the PDF specified by $f_*(x;\lambda)=\lambda e^{-\lambda x}$, $\lambda,x>0$ and the cumulative distribution function (CDF) given by $F_*(x;\lambda)=1-e^{-\lambda x}$. In expanded form, $f(x;\alpha,\lambda)$ is expressed as

$$f(x;\alpha,\lambda)=\frac{1+\alpha}{\alpha}\lambda e^{-\lambda x}(1-e^{-\alpha\lambda x}),\quad x>0.$$

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The corresponding CDF is obtained as

$$F(x;\alpha,\lambda) = \frac{1+\alpha}{\alpha}\left[1 - e^{-\lambda x} - \frac{1}{1+\alpha}(1 - e^{-\lambda x(1+\alpha)})\right], \quad x > 0. \tag{1.1}$$

Then, it is shown in [11] that the WE distribution corresponds to a hidden truncation distribution, as well as the distribution of a sum of two independent random variables following exponential distributions with parameters λ and $\lambda(1+\alpha)$, respectively. Also, the importance of the new shape parameter α on the possible shapes of $f(x;\alpha,\lambda)$ is discussed, increasing the limited possibilities of the former exponential PDF. As important feature, the exponential distribution is a limiting distribution of the WE distribution; it is obtained by taking $\alpha \to +\infty$. For data analysis purposes, the WE model reveals to be an interesting alternative to various weighted exponential models, as the gamma, Weibull or generalized exponential models. Further details on the WE distribution can be found in [1], [19], [18], [8] and [6].

The following new remark is at the basis of this study. One can express the CDF given by (1.1) as the following generalized two-component mixture:

$$F(x;\alpha,\lambda) = w_1 F_*(x;\lambda) + w_2 F_*(x(1+\alpha);\lambda), \tag{1.2}$$

with $w_1 = (1+\alpha)/\alpha > 0$ and $w_2 = -1/\alpha < 0$ satisfying $w_1 + w_2 = 1$, and $F_*(x;\lambda)$ and $F_*(x(1+\alpha);\lambda)$ are two valid CDFs, the second one being a scale version of the first one. For more details on the concept of generalized mixture distributions, we refer the reader to [21] and [12]. Motivated by the overall simplicity and great applicability of the WE distribution, our idea is to use the particular mixture structure of (1.2) to construct a general class of lifetime distributions, called the special generalized mixture-generated (SGM-G) class. As prime result, we consider a generic parent lifetime distribution with CDF denoted by $G(x;\xi)$ instead of $F_*(x;\lambda)$, and put the necessary conditions on it such that (1.2) remains a valid CDF. Note that only one extra parameter is introduced in comparison to the parent distribution, with the perspective of a significant gain in terms of statistical modelling. Then, we investigate the essential functions of the new class, and list new lifetime distributions of interest based on the half-Cauchy, half-logistic, half-normal and Lomax distributions, called SGMHC, SGMHL, SGMHN and SGMLx distributions, respectively. The general properties of the class are investigated in terms of those of the parent distribution, discussing distributional results, diverse moments measures including crude moments, variance, index of dispersion, skewness and kurtosis, also incomplete moments, characteristic function and bivariate extensions based on various copulas. As concrete application, these properties are applied to the SGMLx distribution. Then, the inferential issue of the SGMLx model is discussed through the maximum likelihood method. A complete analysis of practical data is performed for illustrative purposes, showing that the fit power of the SGMLx model is competitive in comparison to some existing lifetime models of the literature.

The sections making up the paper are as follows. Section 2 precises the SGM-G class, with description of its main functions and a short list of special distributions. Several properties of the SGM-G class are discussed in Section 3. Section 4 is devoted a data analysis, revealing the applicable potential of the SGMLx model. Some concluding notes are formulated in Section 5.