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All Commuting Solutions of a Quadratic Matrix Equation for General Matrices*

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Abstract Using the Jordan canonical form and the theory of Sylvester's equation, we find all the commuting solutions of the quadratic matrix equation $AXA = XAX$ for an arbitrary given matrix A.

Keywords Jordan canonical form, Sylvester's equation.

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1. Introduction

The purpose of this paper is to determine all the commuting solutions of the quadratic matrix equation

$$AXA = XAX, \tag{1.1}$$

where A is a given $n \times n$ complex matrix. This equation is called the *Yang-Baxter-like matrix equation* since it has a similar pattern to the classical Yang-Baxter equation introduced independently by Yang in [11] and Baxter in [1], which is famous in statistical physics with close relations to knot theory, braid groups, and quantum groups [8, 12].

Finding all the solutions of (1.1) is difficult for general A, and so far it is only possible for some special matrices as in [10]. This is due to the fact that if we multiple out the both sides of the equation, solving it is equivalent to solving a system of n^2 quadratic polynomial equations in n^2 variables, which is a challenging task in general. Thus, the current research on solving (1.1) is mainly focused on finding commuting solutions, namely the solutions that commute with A. Some recent papers have been devoted to finding various commuting solutions of (1.1) with different assumptions on A. In particular, corresponding to each eigenvalue of A, a spectral projection solution was obtained in [3]. When all the eigenvalues of A are semi-simple, the whole set of the commuting solutions of (1.1) has been successfully constructed in [6] with the help of a result on unique solutions of the Sylvester equation.

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A natural question arises: can we find all the commuting solutions of (1.1) if A is not diagonalizable? A serious study about it began with the paper [7] in which all the commuting solutions have been described when A is a general nilpotent matrix, based on the above mentioned result on the Sylvester equation and the structure theorem of [2, 13] on matrices that commute with a Jordan block with eigenvalue zero.

In this paper, based on the ideas developed in the above works, we want to extend the main result of [7] from a nilpotent matrix to an arbitrary one. We shall give a general solution structure theorem on all the commuting solutions of (1.1), thus giving an answer to the question of finding all commuting solutions of a general Yang-Baxter-like matrix equation. After the paper was written up, we learnt that the same problem was also studied in a recent paper [9] with a different approach. In the next section we present some key lemmas for our purpose, and the main result will be given in Section 3. Some concrete examples constitute in Section 4 to illustrate our theorem, and we conclude with Section 5.

2. Preliminaries

Let A be an arbitrary $n \times n$ complex matrix. The following lemma provides an equivalent way to solve (1.1) for commuting solutions, which was proved in [7].

Lemma 2.1. *A matrix X satisfies $AX = XA$ and $AXA = XAX$ if and only if $AX = XA$ and $X(X - A)A = 0$.*

As proved in [4] (Lemma 3.1), solving (1.1) for a given matrix A is equivalent to solving a simpler Yang-Baxter-like matrix equation

$$JYJ = YJY, \tag{2.1}$$

where $J = U^{-1}AU$ is the Jordan form of A, and the solutions X to (1.1) and the solutions Y to (2.1) satisfy the relation $X = UYU^{-1}$. So from Lemma 2.1, we just need to solve the system

$$JY = YJ,\ Y(Y - J)J = 0$$

to find all the commuting solutions of (2.1). Then all the commuting solutions to (1.1) are given by $X = UYU^{-1}$.

Denote

$$J_j(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & \lambda & 1 \\ 0 & 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}$$

the $j \times j$ Jordan block with eigenvalue λ. In particular, the Jordan block $J_j(0)$ corresponding to eigenvalue 0 satisfies $J_j(0)^j = 0$. The following lemma is a generalization of Theorem 5.15 of [2] from eigenvalue zero to any eigenvalue, but its proof is basically the same and is included for reader's convenience.