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Existence of Positive Solutions for a Nonlinear Second Order Periodic Boundary Value Problem*

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Abstract By using the first eigenvalue corresponding to the relevant linear operator and the topological degree theorem, sufficient conditions for the existence of positive solutions for a nonlinear second order periodic boundary value problem are given. Our results improve and generalize some preliminary works.

Keywords Positive solutions, First positive eigenvalue, Green's function, Topological degree.

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1. Introduction

In recent years, due to the widespread applications in the field of physics and engineering, the study of the existence of the positive solutions for second-order differential equations has attracted the attention of many scholars [2, 9, 11].

In [12], Nieto studied the periodic boundary value problem for the second order differential equation

$$\begin{cases} -x'' = f(t, x(t)), \quad t \in [0, 2\pi], \\ x(0) = x(2\pi), \quad x'(0) = x'(2\pi), \end{cases}$$

where f satisfies Carathéodory conditions. Their main method is the upper and lower solutions.

In [13], by using the Krasnoselskii fixed point theorem, Torres obtained the existence of solutions to the following periodic boundary value problem

$$\begin{cases} x'' = f(t, x(t)), \quad t \in [0, T], \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases}$$

where f is also a function of L^1-Carathéodory type and T-periodic in t.

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In [4], Jiang studied the existence of the positive solutions to the following equation

$$\begin{cases} x'' + Mx = f(t, x(t)), \quad t \in [0, 2\pi], \\ x(0) = x(2\pi), \quad x'(0) = x'(2\pi), \end{cases}$$

where $f \in C([0, 2\pi] \times \mathbb{R}^+, \mathbb{R}^+)$ and $M > 0$. The main method is Krasnoselskii fixed point theorem.

For the following periodic boundary value problem

$$\begin{cases} x'' + a(t)x = f(t, x(t)), \quad t \in [0, T], \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases} \tag{1.1}$$

when f is nonnegative, Li [8] obtained the existence of positive solutions for Eq.(1.1) by using the Krasnoselskii fixed point theorem, Li and Liang [7] also established the existence of the positive solutions for Eq.(1.1) by using the fixed point index theory on a cone. Moreover, in [10], the authors investigated the existence of the positive solutions for Eq.(1.1) under the condition that f may take negative values and the nonlinearity may be sign-changing.

Motivated by the above papers, in this paper, we study the existence of the positive solutions for the following second order periodic boundary value problem

$$\begin{cases} x'' + h(t)\,x' + a(t)x = g(t)f(t, x), \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases} \tag{1.2}$$

where $h \in C([0, T], \mathbb{R}^+)$, $a \in C([0, T], \mathbb{R}^+)$ and $a \not\equiv 0$, $g \in C((0, T), \mathbb{R}^+) \cap L[0, T]$ and $\int_0^T g(t)dt > 0$, $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$, in which $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$. In particular, the function g may be singular at $t = 0$ or $t = T$, f may take negative values and the nonlinearity may be sign-changing. Moreover, when $h(t) \equiv 0$, $g(t) \equiv 1$, Eq.(1.2) becomes Eq.(1.1).

Three highlights should be pointed out. The damping term $h(t)x'$ has been added to generalize the previous equations, g may be singular at $t = 0$ or $t = T$ and f can take negative values and be sign-changing.

The paper is organized as follows. Some useful lemmas for the proof of the main results are given in Section 2. The main results will be given and proved in Section 3. Two examples are given to support our main results in Section 4.

2. Preliminaries

We say the linear system

$$x'' + h(t)\,x' + a(t)x = 0, \tag{2.1}$$

associated to periodic boundary conditions

$$x(0) = x(T), \quad x'(0) = x'(T) \tag{2.2}$$